DUNBAR'S INDUCTIVE ARITHMETIC.

DUNBAR'S
Inductive Arithmetic

A Presentation of Business Methods Based upon Inductively Developed Principles.

By J. H. Dunbar, A. M., Claremont, N. H.

Published in Four Parts, Parts I and II now ready,
Parts III and IV in preparation.

The following are the two points of special superiority claimed for this work:

It develops arithmetical principles, in place of presenting arbitrary arithmetical rules; it thus stimulates the interest and expands the mind of the pupil.

It aims to present direct, common-sense, business-like methods, and thus to produce correct and rapid accountants.

That these claims are not without some foundation will, we think, be admitted by those who read the testimonials in this circular from prominent business men and well-known educators and school officials.

An idea of the plan followed in the development of principles may be obtained from the sample page on the back of this circular. Such development of a principle is always followed by model solutions, complete and systematic explanations, and a large number of exercises for practice.

THE HOWE PRESS, GRANTHAM, N. H.

Hartford, Vt., Aug. 8, 1894.

Prof. J. H. Dunbar,

Plain of comprehension

Dear Sir: I have carefully examined your "Inductive Arithmetic," and find it more than interesting.

It presents a method of instruction for the school room which is at once original, clear, and plain of comprehension to the student mind. Its arrangement is methodical, building on from one branch to another in logical sequence, so as to present a truly inductive system of instruction.

A truly inductive system

Its propositions and definitions are terse and come directly to the point. From the business man's standpoint it also has superior suggestions and practical methods. Its ready ways of calculation in every-day transactions, and particularly in matters of interest computations, are quick and economize one's time. A complete mastery of this work by the student would seem to equip him for all the arithmetical problems of commercial and ordinary business life. Your treatise deserves success and will command it.

Comes directly to the point

Truly yours,

(Ex-Governor of Vermont, and President of White River Saving Bank) Samuel E. Pingree.

A new departure

Bethel, Vt., June 28, 1894.

J. H. Dunbar, Esq.,

Dear Sir: I have recently examined a copy of your Arithmetic with much interest. It is a new departure in the line of text-books. The arrangement is admirable, and the style clear, simple, and logical. It seems calculated to stimulate the mind of the pupil and suggest practical methods to the instructor; to educate as well as to instruct.

Educates as well as instructs

The treatment of the subject of interest is the best I have ever seen, accurate and plain.

A universal benefit to scholars and schools

The general adoption of the book is to be hoped for and will be a universal benefit to scholars and schools. I have no doubt it will receive the practical approval it deserves.

Sincerely yours,

(Ex-Pres. Vt. Bar Association.) Wm. B. C. Stickney.

Burlington, Vt., Aug. 1, 1894.

Presents practical business methods

Mr. J. H. Dunbar,

My Dear Sir: I have examined your new "Inductive Arithmetic," and am much pleased with it. You seem to have caught and confined in your book practical business methods. This is emphatically true of those portions devoted to "Compound Num-

bers" and to "Interest." I congratulate you on your success in your new field.

Very truly,

(Member of Congress from Vermont.) D. J. Foster.

Remarkable for short cuts and logical reasoning

Beverly, Mass., July 25, 1895.

Mr. J. H. Dunbar,

Dear Sir:

I have examined your "Inductive Arithmetic" with keen interest. It is remarkable for the clearness of its methods, its short-cuts to results, and the way in which from first to last the pupil is led to reason logically.

Effects quick, correct solutions

Because its methods thus effect quick, correct solutions, and at the same time tend to develop the reason, it meets equally the demand both of the business man and the teacher.

The pupil must receive from its study that intellectual stimulus which he fails to get from those arithmetics which merely frame rules for purely mechanical processes.

Meets demand both of business man and teacher

Yours cordially,

Prin. Beverly (Mass.) Training School. Isabel Chapin.

Claremont, N. H., Mar. 24, 1902.

During the spring of 1901, while I was a member of the Claremont School Board, portions of Dunbar's Inductive Arithmetic, Part I. and "Business Methods in Interest," were introduced into the Claremont schools. From personal knowledge I can testify that both pupils and teacher were highly pleased with the books, and that the results of their use were in every way satisfactory. To those school officers who, like myself, believe that the best text-book on arithmetic is the one which presents the most business-like methods, and which best develops thinking on the part of the pupil, I heartily commend this work.

Highly satisfactory to teachers and pupils

C. H. Wilson.

Unequaled for supplementary work

Lempster, N. H., Mar. 26, 1902.

This is to certify that we used Dunbar's Inductive Arithmetic in three of our schools last fall and liked them very much. We never have had any book so good for supplementary work or one which awakened the reasoning faculties in children so well. I should recommend its use in any school advanced enough to use it.

Unrivaled for awakening reasoning faculties

Jennie L. Olmstead, Member of School Board.

Each question is to be read, weighed, and answered by the pupil, who thus gains practice in expressive reading and in clear conception and expression of ideas

39. To Multiply by 14 7, 125 25 5, etc.

We are to multiply a certain number by 147.
We first multiply by 7 units.
We have left 14 tens by which to multiply.
How will the product by 14 compare with the product by 7?
How, then, after obtaining the product by 7 can we obtain the product of the same multiplicand by 14?

If the product by 7 is 28, what will be the product by 14?
If the product by 7 is 91, what will be the product by 14?
If the product by 7 is 214, what will be the product by 14?

How will the order of a product by tens compare with the order of the product of the same multiplicand by units?
Where, then, should the right-hand figure of the product by the 14 tens be placed with reference to the right-hand figure of the product by the 7 units?
Give, then, a special rule for multiplying by 147.

We are to multiply a number by 125 25 5.
We first multiply by 5.
After obtaining the product by 5, how can we obtain the product by 25?
Where shall we place the right-hand figure of the product by the 25 tens?
How will the product by 125 compare with the product by 25?
How, then, after obtaining the product by 25, can we obtain the product by 125?

How does the order of the 125 compare with the order of the 25?
How, then, will the order of the product by 125 compare with the order of the product by 25?
Where, then, should the right-hand figure of the product by 125 be placed with reference to the product by 25?
Give, then, a special rule for multiplying by 125255.

Where should the right-hand figure of a partial product be placed with reference to the right-hand figure of a preceding partial product?
If the partial multiplier is four orders higher than the preceding partial multiplier?
If it is five orders higher?
If it is six orders higher?
If it is ten orders higher?

Give, then, a special rule for multiplying a number

By 42 6.
By 6 42.
By 36 6.
By 21 7.
By 7 21.
By 600 150 30 6.
By 625 125 25 5.
By 256 128 32 8.
By 210 105 15 5.
By 3 9 27 81.

AN INDUCTIVE ARITHMETIC

FOR

INTERMEDIATE AND HIGHER GRADES OF PUBLIC AND PRIVATE SCHOOLS

BY

J. H. DUNBAR, A. M.

CLAREMONT, N. H.:
PUBLISHED BY J. H. DUNBAR
1902

Preface.

Shall arithmetic be taught as a science or as an art? Shall its principal aim be to aid in developing a trained and logical brain or to produce an expert accountant? It is the theory of the author of this book that the subject should be regarded as equally important from both stand-points, and that both aims should be kept in view alike by those who write text-books and those who teach them. In spite of the fact that the statement is that of one who has written an arithmetic issued by a large publishing house, and of one who is adorned with the degree LL. D., it is rather less than a half-truth to assert that "In the extending of taxes, or the finding of interest in banks, or in making bills for lumber, for excavations, for mechanics' work, etc.—the results are not obtained by computation; they are taken from prepared tables." The statement may apply in certain sections of our country, but in New England, at least, the carpenter is as little troubled at being called upon to make out a bill of lumber as he would be at being required to repeat the easier portions of the multiplication table, and the bank cashier or clerk names the interest on any ordinary principal, for 60 days, or any convenient part or multiple of that number, in less time than he could open the pages of the book with prepared tables. What characterizes the real student of arithmetic is not a pocket filled with printed technical tables, but a head filled with thoroughly comprehended principles, and with the necessary number, always small, of essential facts. A young

man thus endowed need have no anxiety in whatever field of arithmetic work he may chance to be placed. As may be assumed from the preceding remarks, the author will endeavor in the remaining parts of his arithmetic to give a clear and comprehensive treatment, with numerous concrete illustrations, of practical measurements and the problems of commercial arithmetic.

The plan of this book may be stated in a few words. It is to guide the mental efforts of the pupil instead of performing the work for him, and to guide them along the most direct and natural path, whether this be the beaten track followed through many generations, or some hitherto unexplored way which serves to straighten and shorten a crooked portion of the regular road. In other words the two-fold aim kept constantly in view is that the pupil should follow the most direct path, and know his exact bearings at every point. To make the attainment of this end absolutely certain, every page of the book has been developed and tested in the class room, and every principle has been demonstrated to be within the ready comprehension of any average pupil. That it will be as favorably received by the progressive teachers of New England, and prove as satisfactory to them, as the part that preceded it, is the hope and confident belief of its author.

Contents.

An Inductive Arithmetic.

Notation and Numeration.

A certain table is 12 feet long. A second table is 4 feet long. The length of the first table is how many times that of the second?

The surface of the first table is 60 square feet, and of the second table 12 square feet. The surface of the first table is how many times that of the second?

The volume of a certain block of wood is 8 cubic feet. The volume of a second block is 2 cubic feet. The volume of the first block is how many times that of the second?

The weight of the first block is 500 pounds, and of the second block 100 pounds. The weight of the first block is how many times that of the second?

A boy can walk a certain distance in 25 minutes. He can ride the same distance on his bicycle in 5 minutes. His speed on the bicycle is how many times his speed on foot?

A man pays for a carriage $90 in ten-dollar bills. How does the value of the carriage compare with the value of each of the bills?

A certain bridge will sustain a weight of 20 tons. A second bridge will sustain a weight of only 5 tons. The strength of the first bridge is how many times that of the second?

Anything that can be measured, or definitely compared, is called **Quantity**. Observe that the quantities we have compared are,

In the first problem, extensions in one direction, or Length.

In the second problem, extensions in two directions, or Surfaces.

In the third problem, extensions in three directions, or Volumes.

In the fourth problem, the comparative effects of the attraction of the Earth on two bodies, or Weights.

In the fifth problem, the durations of Time required by the two methods of locomotion.

In the sixth problem, the Values of the two articles, as determined by the relative desires for each of them by the buyer and seller.

In the seventh problem, the Capacities of the two bridges to resist the effect of gravity on bodies supported by them.

NOTE 1. The first six quantities, Length, Surface, Volume Weight, Time and Value, are the ones most commonly met with in arithmetical computations. Among the other capacities which can be measured by their effects, are the strength of an electric current, which is measured in volts and amperes; the power of a steam-engine, which is measured by horse-powers, and the intensity of lights, which is measured by candle-powers.

NOTE 2. It might seem necessary to include gallons, bushels, etc., in a separate class. A little reflection, however, will show that these quantities fall under one of the first three classes. Under which class do they fall and why?

About how long is your school-room?

About what quantity of boards is there in its floor?
(To measure a surface multiply its length by its breadth.)

About what quantity of air does it contain?
(To measure a volume multiply together length, breadth, and thickness.)

About what is the weight of this air?
(It takes about 12 cubic feet of air to weigh a pound.)

About how long does it take you to walk from your home to your school-room?

About what is the value of the taxable property of your town?

What standard quantity, in common use and established by law, do we employ in the first of the preceding problems? in

the second? in the third? in the fourth? in the fifth? in the sixth?

A quantity employed to measure another is called the **Unit** of the measured quantity, and the process of measuring consists in finding, directly or indirectly, how many, or the **Number** of, times the unit can be applied. Hence the following definitions:

Quantity is anything that can be definitely measured.
(Name four quantities.)

A Unit of Quantity is a quantity employed as a standard of measurement.
(Name the unit that would be employed in measuring each of the four quantities you have selected.)

Number is the answer to the question, How many?
(Name the number you would obtain in measuring each of the four quantities you have selected.)

Arithmetic is the science of numbers.

Numbers of One Figure.

1. Numbers to ten are expressed in Arithmetic as follows:

	one	two	three	four	five	six	seven	eight	nine
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

The characters 0, 1, 2, 3, 4, etc. are called **Figures**; the numbers they represent when standing alone are called **Units**. The character **0**, which represents nothing in itself, is called **Zero**, and, in distinction, **1**, **2**, **3**, etc., which signify value though standing alone, are called **Significant Figures.**

The process of expressing numbers in figures is called **Notation.** When numbers are expressed in figures, the process of reading them or writing them in words is called **Numeration.**

Numbers of One Period.

2. Groups of units from nine through nine hundred ninety-nine, and parts of units from one thousandth through nine hundred ninety-nine thousandths, are expressed in accordance with the following

PRINCIPLES.

Each Removal of a Figure one Place

To the left increases its value tenfold.	To the right decreases its value tenfold.
Thus, 6 signifies 6 units; but 6 0, six tens; and 6 0 0, six hundreds.	Thus, 6 signifies six units; but 0.6, six tenths; 0.06, six hundredths; and 0.0 0 6, six thousandths.

NOTE 1. The place a figure occupies is called its ORDER. As the value of a figure depends upon its position relative to a unit figure, it follows that an order lacking a significant figure must be represented by a zero.

Write in figures eight hundred four, thirty, five hundred six, seven hundred.

NOTE 2. In writing six tenths, eight hundredths, etc.,it is not customary to write a 0 in units' order. A period, however, must always be placed at the left of tenths' order.

Thus, six tenths is commonly written not 0.6, but .6; and eight hundredths, not 0.08, but .08.

Write in figures seventeen hundredths, six thousandths, five hundredths, fifty-four thousandths.

The period when thus written is called the DECIMAL POINT, and the numbers at its right a DECIMAL. In distinction, numbers at the left of the point are called INTEGERS, or Whole Numbers. It will be observed that the period marks the end of the whole number, just as in its ordinary use it marks the end of a sentence.

NOTE 3. In reading a number of three integral orders, the name "units" is omitted; "hundred" is used instead of "hundreds," and two tens, three tens, etc. are read "twenty," "thirty," etc. as explained in the Appendix.

Thus, 235 is read two hundred thirty-five.

Write in words 864, 327, 906.

NOTE 4. An order represented by a 0 is not named.

Thus, 805 is read eight hundred five.

Write in words 708, 405, 310.

NOTE 5. To read a decimal of one, two, or three orders, read as in integer, and add the name of the right-hand order.

Thus, .2 is read two tenths; .02, two hundredths; .002, two thousandths; .642, six hundred forty-two thousandths.

Write in words, .708, .56, .709, .84, .084, .007.

NOTE 6. It will be observed that the name of each decimal order is formed from the name of the corresponding integer by adding "th."

Thus, ten, ten-th; hundred, hundred-th; thousand, (see Art. 3), thousand-th.

NOTE 7. As shown in Note 3, in writing a number in words a hyphen is written between tens and units.

Thus, 35 is written thirty-five.

Write in words 76, 84, 33, 27.

NOTE 8. A number that consists of both integers and decimals is called a MIXED NUMBER. In writing a mixed number in words,

(1) Place a comma after the integral part of the number.

(2) Connect the integral and the decimal parts by the conjuuction "and."

Thus, 3.4 is written three, and four tenths; 506.93 is written five hundred six, and ninety-three hundredths.

Write in words, 75.43, 804.96, 706.003.

NOTE 9. Each group of three orders, in integers beginning with units and in decimals beginning with tenths, is called a PERIOD. The left-hand period of integers and the right-hand period of decimals, however, may consist of only one or two figures.

Thus, 854 is a number of one period; .756 is a number of one period; but 2.9 is a number of two periods.

Ex. 1.

Write in words the following numbers:

(1)	(2)	(3)	(4)	(5)	(6)
2	23	353	575.5	749.97	856.868
27	359	699	63.98	999.897	599.885
398	379	509	8.094	697.429	575.975
.3	2.3	53	85.076	36.855	708.989
.03	.26	2.79	752.89	983.567	909.454
.003	.259	.023	787.543	350.636	835.737
39.237	36.087	525.527	460.747	3.842	492.863
95.938	505.753	90.392	905.274	255.032	929.562
86.645	83.573	807.423	735.053	575.231	22.482
98.357	402.978	52.808	835.992	953.624	907.283
720.820	894.929	853.578	892.808	625.802	892.362
75.382	850.024	386.372	935.725	952.074	345.256
973.784	97.353	52.582	545.536	824.306	752.37
988.877	382.688	899.887	937.872	223.788	877.867
290.867	594.754	789.909	985.723	803.073	959.352
652.687	332.223	267.576	864.473	323.567	722.524

Ex. 2.

Write in figures the following numbers:

1. The numbers from one to thirty; from one tenth to nine tenths; from nine hundredths to thirty hundredths; from ninety-six thousandths to one hundred three thousandths.

2. Seven hundred; seventy; seven tenths; seven hundredths; seven thousandths.

3. Two hundred forty-eight; twenty-four, and eight tenths; two, and forty-eight hundredths; two hundred forty-eight thousandths.

4. One hundred fifteen thousandths; fifteen thousandths; five thousandths; five hundredths; five tenths.

5. Ninety-seven; ninety; ninety hundredths; nine tenths.

6. Five hundred sixteen, and eighty-four hundredths; forty-two, and sixteen hundredths; one hundred six, and nine tenths; seven hundred three, and nine thousandths.

7. One hundred twenty-three; nine hundred five; seven hundred; five hundred thirty.

8. One hundred eleven, and four hundredths; eight hundred, and six tenths; ninety-nine, and thirteen thousandths.

9. Five hundred one; five hundred ten; one hundred nine; seventeen hundredths; seven hundredths.

10. Six hundred two; nine hundred fifteen; three hundred seven; fourteen thousandths.

11. Eight hundred ninety-four, and seven hundred seventy-three thousandths; five hundred twenty-eight, and three hundred eighty-four thousandths; two hundred one, and seventy hundredths; eight, and eight hundredths.

12. Eight hundred ninety-three, and sixty five thousandths; four hundred seven, and nine hundredths; two hundred five, and eight hundred seventy-six thousandths; eight hundred one, and forty-nine thousandths; nine hundred thirty-seven, and twenty-three thousandths.

13. One hundred ninety-seven, and two thousandths. Five hundred three, and eight hundredths; seven hundred eight, and thirty-nine thousandths; seven hundred eighty-two, and nine hundred six thousandths.

Numbers of More Than One Period.

3. Integral or decimal numbers requiring more than three figures may be expressed by combining with the preceding principles the following additional

PRINCIPLES.

Each Removal of a Period Three Places

To the left increases its value a thousandfold.

Thus, 123 signifies 123 units; but 123 000, 123 thousands; and 123 000 000, 123 millions.

To the right decreases its value a thousandfold.

Thus, 123 signifies 123 units; but .123, 123 thousandths; and .000 123, 123 millionths.

Trillions.	Billions.	Millions.	Thousands.	Units.	Thousandths	Millionths	Billionths	Trillionths
806	000	124	005	018.	324	560	080	928
	475	307	856	754.				
				.	389	471	528	754
111	222	333	444	555.	666	777	888	999
999	888	777	666	555.	444	333	222	111
436	000	000	000	000.	000	000	000	436
	918	870	043	127.	018	720	001	5
1	888	263	049	173.	144	303	086	65

NOTE 1. We have learned that the value of a period depends upon its position relative to the decimal point. It follows, therefore, that a period lacking a significant figure must be represented by three zeros.

Thus, eight million three hundred twenty-four is written 8000 324.

NOTE 2. In reading, observe the following points:

(1) The unit period is not named.

Thus, as already explained, 456 is read four hundred fifty-six.

(2) The names of the other integral periods are considered to be adjectives agreeing with the word "units" understood, and are, therefore, as shown in Note 1, singular in form.

Thus, 123 000 is read one hundred twenty-three thousand, instead of one hundred twenty-three thousands.

(3) The names of the decimal periods are considered to be nouns, and are read as singular or plural according to the nature of the period.

Thus, .001002 is read one thousandth two millionths.

(4) A zero period is not named.

Thus, 13 000 024 is read thirteen million twenty-four.

NOTE 3. If a decimal is not made up of complete periods, the figure or figures remaining at its right are read as tenths or hundredths of the last preceding period.

Thus, .123 4 is read 123 thousandths 4 ten-thousandths; .123 45, 123 thousandths 45 hundred-thousandths; and .123 456 7, 123 thousandths 456 millionths 7 ten-millionths.

The expressions "hundred-thousandths," "ten-millionths," etc. are equivalent to "hundredths of thousandths," "tenths of millionths," etc.

NOTE 4. The orders of the other periods have the same relation to each other as those of units' period.

Thus, the orders of the second number in the diagram are

Units	Thousands	Millions	Billions
Tens	Ten-thousands	Ten-millions	Ten-billions
Hundreds	Hundred-thousands	Hundred-millions	Hundred-billions

and the orders of the third number are

Tenths, hundredths, thousandths; ten-thousandths, hundred-thousandths, millionths; ten-millionths, hundred-millionths, billionths; ten-billionths, hundred-billionths, trillionths.

Instead of the terms "ten-thousands," "hundred-millions," etc. the uncontracted terms "tens of thousands," "hundreds of millions," etc. are frequently used.

NOTE 5. Distinguish carefully between such decimals as three hundred thousandths and three hundred-thousandths. In writing these decimals in words, the distinction is easily made by the proper use of the hyphen; in reading them aloud, make the distinction by the inflection of the voice and the relative length of the pauses between their several parts.

Thus, the first number in this note should be read three hundred thousandths; the second, three hundred-thousandths.

NOTE 6. It will be observed that Note 6, Article 2, applies to all decimal orders.

Thus, ten-thousand, ten-thousand-th; hundred-million, hundred-million-th.

NOTE 7. An integral number of four figures may be read as hundreds and units.

Thus, 7854 may be read seventy-eight hundred fifty-four.

This method is especially followed in reading dates. Thus, "the year 1901" is read "the year nineteen hundred one."

NOTE 8. As stated in Note 8, Article 2, in writing a mixed number, the conjunction 'and' is used to connect the two parts. In this book, neither the comma nor the conjunction is used between any other parts of a number. For a general discussion of the use of the comma and the conjunction in this connection, see the Appendix.

NOTE 9. For the names of the periods above trillions and below trillionths, see the Appendix.

NOTE 10. Applying the principles explained in the preceding notes, we thus read the first number in the diagram:

Eight hundred six trillion one hundred twenty-four million five thousand èighteen, and three hundred twenty-four thousandths five hundred sixty millionths eighty billionths nine hundred twenty-eight trillionths.

Ex. 3.

Write in words each of the numbers in the preceding diagram.

Ex. 4.

Construct a diagram similar to that given in Article 6. Place above each column the name of the period to be written in it, and write in their appropriate columns the following numbers:

1. Three hundred twelve trillion eight hundred seventy-six billion nine hundred sixty-seven million three hundred eleven thousand six hundred sixteen, and seven hundred twenty-four thousandths eight hundred eighty millionths four hundred twenty-six billionths one hundred forty-six trillionths.

2. Eight trillion fourteen million fifty, and four hundred thirty-seven thousandths nine millionths eighty-three hundred-millionths.

3. One hundred seven trillion eight hundred seventy-nine million thirty-one thousand, and one hundred seventy-one thousandths.

4. Eight hundred eleven billion two hundred twenty-seven, and eight hundred ninety-four thousandths seven hundred millionths seven hundred-millionths.

5. Two hundred forty-three trillion five hundred twenty-seven billion thirteen thousand two hundred fifty-four.

6. One million four hundred forty-eight, and nine hundred thirty-one billionths twenty-one hundred-billionths.

7. Ninety-nine million seven hundred thousand two, and eighty-five thousandths seven hundred billionths seven ten-billionths.

8. Nine hundred fifty-four trillion, and nine hundred fifty-four trillionths.

9. Two hundred seventy trillion two hundred seventy million two hundred seventy, and two hundred seventy millionths two hundred seventy trillionths.

10. Nine trillion nine billion nine million nine thousand nine, and nine thousandths nine millionths nine billionths nine trillionths.

11. Three trillion five billion one million three, and seven thousandths.

12. Seventeen, and seven tenths; seventy, and seven hundredths; seven hundred, and seven thousandths.

13. Seven trillion seven billion, and seven millionths seventy billionths nine hundred-trillionths.

14. Four hundred eight million thirty-five, and five hundred six millionths seven hundred-billionths.

15. Eighty-three trillion five hundred six billion thirteen thousand twenty-seven, and three hundred fifteen thousandths seven millionths two hundred billionths two hundred-billionths.

16. Twenty-three trillion one hundred eight million seven hundred eighteen thousand six, and seven thousandths eight hundred four millionths seven hundred thirty trillionths eight hundred-trillionths.

17. Seventeen billion eight thousand thirty-four, and seventeen millionths four ten-billionths; eight trillion seventeen million five thousand twenty-four; seven hundred thousandths seven hundred-thousandths.

Ex. 5.

123	123	123	123	123.	123	123	123	123
	75	307	764	820.	913	034	000	8
				.	000	004	370.	92
720	913	726	804	622.	804	875	012	006
133	748	794	007	535.	143	000	801	111
	717	228	178	609.	844	159	293	95
750	060	000	510	007.	011	122	8	
	481	175	052	302.	086	943	518	5

NOTE 1. Before attempting to read the numbers in the preceding diagram, learn in order the names of the periods from trillions to trillionths.

NOTE 2. The first step in reading a number in the diagram is to ascertain the name of its left-hand period. To do this,

(1) If the number is an integer or a mixed number, begin with units' period and name in order the periods to the left.

Thus, in the second number in the diagram, "Units, Thousands, Millions, Billions." The number, therefore, is 75 billion 307 million, etc.

(2) If the number is a decimal, begin with thousandth's period, and name in order the periods to the right until the name of the first period containing significant figures has been ascertained.

Thus, in the third number in the diagram, "Thousandths, Millionths." The number, therefore, is 4 millionths, etc.

In thus finding the name of the left-hand period, do not name the periods aloud. Practise this mental naming until with any ordinary number you can name the left-hand period at a glance.

Ex. 6.

Construct a diagram similar to that given in Exercise 5, and in it write the following numbers:

1. One hundred fourteen thousand sixty-five, and thirteen thousandths twenty-four hundred-thousandths.

2. Eight trillion eighty million eight hundred thousand eight.

3. Ninety-three thousandths seven billionths eighteen hundred-billionths.

4. Seven hundred twenty-seven trillion one billion eleven million one hundred eleven thousand, and eight hundred thirty billionths.

5. Twenty thousandths seven hundred eight millionths, twenty-seven billionths.

6. Five billion seventeen.

7. Forty-six millionths seven hundred-millionths.

8. Seven million thirty-eight thousand seven hundred forty-one, and six thousandths sixty millionths seven ten-millionths.

Figures Separated into Periods.

4. In printing figures it is customary to separate them into periods by the use of commas. Follow this system in writing the numbers under Exercises 8, 9, and 10.

Ex. 7.

278,754,905.025,834,55	29,084,320,000.075,876,32
560,074,000.295,007,16	907,000,352.762,190,07
7,000,005,037.007,327,7	9,925,042,752.988,000,608,9
19,075,058.296,720,965,9	18,655,269,267.829,007,4
96,000,000.205,039,700,3	27,000,000,000.254,028
25,909,557.752,400,557,8	74,528,709,037.277,277,009

Ex. 8.

Write in figures the following numbers:

1. One; one tenth; one hundredth; one thousandth; one ten-thousandth; one hundred-thousandth; one millionth.

2. Seven thousand four, and fifteen hundred-thousandths.

3. Nine million eighteen, and three hundred thousandths.

4. Eighteen million four thousand two hundred seven, and three hundred-thousandths.

5. Eight million seventy, and one thousandth eight hundred four millionths seven ten-billionths.

6. Forty-three trillion eighteen million seventy, and seven ten-thousandths.

7. Fifteen million nineteen thousand four hundred, and seven ten-thousandths.

7. Fifteen million nineteen thousand four hundred, and eighteen hundred-thousandths.

8. Sixty billion seventy-three, and twelve millionths.

9. Ninety-four million eleven thousand ten, and eight hundred thousandths seven millionths.

NOTE. To write a period correctly, two conditions must be fulfilled:

(1) Each period, except the first in integers and the last in decimals, must contain three figures.

(2) These figures must be so arranged as to express the correct number.

Thus, let it be required to write seven million sixteen thousand four hundred twenty-six. We first write the figures 7 16 426, separating them into periods by commas as shown below:

7, 1 6 ,426

As millions is the first period, the number of figures it contains is immaterial. Writing 16 in thousands' period, we have in that period only two figures instead of three, the required number. A zero, therefore, must be used in this period. But where shall it be placed? If after 6, the period will read "one hundred sixty;" if between 1 and 6, it will read "one hundred six." Therefore, neither arrangement is correct. But let the 0 be placed before 1 6 , and the period will at the same time contain three figures and read correctly.

Ex. 9.

Write in figures the numbers in Exercise 4.

Ex. 10.

Write in figures the numbers in Exercise 6.

Figures not Separated into Periods.

5. In performing operations with figures it is not customary to separate them into periods by any device.

Ex. 11.

Write in words the following numbers:

197234.80765	389005.140706	43000.0006050
4378.920076	5238409.2207	56070054.32
4056789.0004056	1234567.807	9600438.278
16171800.900087	1920.834545	975321.18675
2000050.0208009	4567002.89074704	6000845.006500328

Ex. 12.

Write without diagram or commas the following numbers:

1. Five million six hundred thirty-four thousand seventeen.

2. Nine thousandths seventy-eight hundred-thousandths.

3. Sixteen million twenty-nine thousand four, and thirty-seven hundredths.

4. Thirty-seven thousand, and two hundred twenty-three thousandths.

5. Four hundred fifty-six thousand two hundred seventeen, and three hundred eight thousandths.

6. Six thousand eight hundred four, and two thousandths four ten-thousandths.

7. Fifteen, and four hundred thousandths four hundred-thousandths.

8. Nine thousand nine, and six thousandths sixty millionths.

9. Seven million eight hundred four thousand six hundred two.

United States Money.

6. The unit of United States money is the **Dollar.** In connection with figures, in place of the word "dollars" the sign $ is used. In reading, hundredths of dollars are commonly referred to as **Cents.**

Thus, $125.35 is read "one hundred twenty-five dollars, and thirty-five cents;" $.64 is read "sixty-four cents;" and $.01 is read "one cent."

Ex. 13.

Write in words the following numbers:

(1)	(2)	(3)	(4)	(5)
$605.07	$9401.36	$35963.53	$459376.05	$7823416.25
$182.50	$8772.97	$11861.00	$238750.93	$1661791.73
$160.17	$6017.41	$80607.96	$170769.47	$5676459.24
$289.75	$8417.61	$52000.23	$171762.71	$7935633.33
$324.02	$6187.33	$17339.95	$216008.66	$2661818.61
$491.10	$7188.28	$30713.41	$907210.98	$4130940.83
$202.27	$1655.10	$64289.63	$532726.58	$6911483.50
$154.42	$2785.08	$57342.13	$615762.30	$1354467.52
$436.04	$5344.87	$83377.22	$298148.64	$1312611.31

Ex. 14.

Write in figures the following numbers:

1. One dollar; ten dollars; one hundred dollars; one thousand dollars; ten thousand dollars.

2. Ninety-nine dollars; ninety dollars; nine dollars; one cent.

3. Ten dollars, and ten cents; one dollar, and one cent; twenty-seven dollars, and twenty-seven cents.

4. Eight thousand one dollars; one cent; eleven thousand eleven dollars, and eleven cents.

5. One hundred five thousand three hundred six dollars, and seventy-four cents.

6. Two million eighteen thousand five dollars, and one cent.

7. Thirteen million one hundred thirteen thousand three dollars, and thirteen cents.

8. Nine million twenty-four dollars, and four cents.

9. Three billion twelve thousand eight dollars, and one cent.

10. Fifteen trillion four thousand two dollars.

11. Three hundred nineteen trillion sixteen billion three hundred six million seven hundred eighty-four thousand dollars.

NOTE. It will be observed that in the two preceding Exercises we have referred to $605.07, one dollar, etc. as numbers instead of quantities. The former term would seem to be incorrect, but it is customary in arithmetic to refer to a number used in connection with a quantity as a CONCRETE NUMBER, and to allow this term to stand for the Quantity itself.

The signification of the term ABSTRACT NUMBER will be explained under Multiplication and Division.

7. Concise and formal directions for performing an arithmetical operation are called a **Rule.** Thus, for writing numbers of more than one period we have the following

RULE.

Commencing at the left, write each period in its proper position.

See that each period, except the first in integers and the last in decimals, both contains three figures and reads correctly.

Give a rule of like character for reading numbers. State

1. *Into what groups the figures of a number are separated.*

2. *What group is first read, in what order the different groups are read, and how the reading of each compares with the reading of "units" group.*

8. The Common Method of Reading Decimals.

Read 567 as units; as tens and parts of a ten; as hundreds and parts of a hundred.

Read the decimal .2347 as parts of a unit; as tenths and parts of a tenth; as hundredths and parts of a hundredth; as thousandths and parts of a thousandth.

Read the decimal .523,476,2 as thousandths and parts of a thousandth; as ten-thousandths and parts of a ten-thousandth; as hundred-thousandths and parts of a hundred-thousandth; as millionths and parts of a millionth; as ten-millionths.

It is evident that in any number, integral or decimal, any order may be taken as the unit, or standard of measurement. In reading decimals the common method is to consider the right-hand order as the standard, and to read the number exactly as an integer, with the name of the right-hand order added. Thus, .1234567 is read one million two hundred thirty-four thousand five hundred sixty-seven ten millionths. This latter method, in case the decimal is of considerable length, is open to the following objections.

It is much more laborious.

Read .00236668 by both methods, and compare the amount of mental effort required in each.

One gets a less clear conception of the value of a decimal read by it.

Write in words by both methods the decimal just given.

Which expression gives you the more definite idea of the value of the decimal?

In spite of these objections, the fact that this method is in common use renders a complete mastery of it desirable.

NOTE. Read by periods when determining the name of the right-hand figure. Thus, in the decimal .005,207,004,56, thousandths millionths, billionths, hundred-billionths.

Read by this method the decimals in Ex. 11.

REVIEW QUESTIONS.

Define quantity; the unit of a quantity; number; arithmetic. Illustrate each definition.

Count to ten. What are the characters 0, 1, 2, 3, etc. called? the numbers they represent when standing alone? What special name is given to 0? to 1, 2, 3, etc.? Define notation; numeration.

Give the two principles for writing numbers of one period.

What does 6 signify? 60? 600? 0.6? 0.06? 0.006?

What is the customary way of writing 0.6, 0.06, etc.?

What is meant by the order of a figure? Name the orders in 622.79.

What must be done when an order lacks a significant figure?

What name is given to the period when placed between units and tenths? What is the number at its right called? at its left? Compare the use of the period in numbers with its use in sentences.

Write in words 235. Give Note 3. Article 2.

Write in words 805. Give Note 4.

Write in words .2; .02; .42; .002; .642. Give Note 5.

What decimal order corresponds to tens? to hundreds? to thousands? Give Note 6.

Write in words 35. Give Note 7.

Define a mixed number. Write in words the mixed number 3.45 the mixed number 527.93. Explain the use of the comma in mixed numbers; the use of the conjunction.

Define a period. How many periods are there in 854? in .756? in 2.9? Give the two principles for writing numbers of more than one period. Name in order the periods from trillions to trillionths.

Write in figures eight million three hundred twenty-four. Give Note 1, Article 3.

Read 456. Give Note 2, Section 1.

Read 123,000. Give Note 2, Section 2.

Read .001 002. Give Note 2, Section 3.

Read 13 000 024. Give Note 2, Section 4.

Read .123 4; .123 45; .123 459 7. Give Note 3. Explain the expressions "hundred-thousandths," "ten-millionths," etc.

Name the integral orders from units to trillions; the decimal orders from tenths to trillionths. Give Note 4. What terms may be used in place of ten-thousands, hundred-millions, etc.?

Read .300; .000 03. Write in figures three hundred millionths; three hundred-millionths. Give Note 5.

What decimal order corresponds to ten-thousand? to hundred-million? Give Note 6.

How is "the year 1893" commonly read? Give Note 7.

Write in words 1 000 000.000 001. Give Note 8.

What is the first step in reading a number of more than one period? How may the name of the left-hand period be ascertained when the number is an integer or mixed number? when the number is a decimal? What is said about naming periods aloud? How does the reading of each of the other periods compare with the reading of units' period? Read 410 051 316.875 109 6.

With printed figures what use is made of the comma? Read 4,087,320,001,092. Write seven million sixteen thousand four hundred twenty-six. Explain the writing of the second period.

What is said about the writing of figures when performing operations? Write without commas six million fifty thousand four hundred, and thirteen thousandths seven millionths. Read 7908765; 1230570897208.

What is the unit of United States money? With figures, what sign represents the word "dollars"? How are hundredths of dollars commonly read? Read $125.35; $.94; $.01.

Define a rule. Give a rule for notation; for numeration.

Explain the effect upon the position of each figure of a number of moving a decimal point to the left.

Explain the second method of reading decimals.

Read the following decimals by this method: .022673; .2666948; .00007932183997.

* * *

Read .7 .007; .000 7; .000 07; .000 007; .000 000 7; .000 000 07; .000 000 007.

Write three tenths; three hundredths; three thousandths; three ten-thousandths; three hundred-thousandths; three millionths; three ten-millionths: three hundred-millionths; three billionths.

How many units are there in a ten? tens in a hundred? hundreds in a thousand? thousands in a ten-thousand? How many tenths are there in a unit? hundredths in a tenth? thousandths in a hundredth? ten-thousandths in a thousandth? How many units of any order, then, does it take to make a unit of the next higher order? What part, then, is a unit of any order of a unit of the next higher order?

Addition and Subtraction.

A bin contains 30 bushels of grain. It is filled by the addition of 20 bushels more.
How many bushels does it then contain?

In a school-yard 19 boys are playing foot-ball. Three more boys join the game.
How many are then playing?

A farmer has three bins of grain. The first contains 30 bushels, the second 20, and the third as many as the first two.
How many bushels does the third bin contain?

In a school-yard 18 boys are playing base-ball and 22 foot-ball.
How many are playing both games?

There are 31 days in October, 30 in November, and 31 in December.
How many days are there in the 3 months?

9. The quantities to be combined in each of the preceding problems are called **Addends**, the process by which they are combined is called **Addition**, and the result of the combination is called the **Sum** or **Amount**. Hence the following definitions:

Addition is the process of combining two or more numbers.

An *Addend* is a number to be combined with other numbers by addition.

A *Sum*, or *Amount*, is the result of an addition.

* * *

A bin contains 50 bushels of grain. Twenty bushels are taken out.
How many bushels remain?

In a school-yard 22 boys are playing foot-ball. Three boys leave the game.
How many remain?

A farmer has three bins of grain. The first contains 50 bushels, the second 30, and the third a quantity equal to the difference between the quantities in the first two.
How many bushels does the third bin contain?

A bin will hold 50 bushels of grain. There are in it 30 bushels.
How many bushels must be added to fill it?

In a school-yard 18 boys are playing base-ball and 22 foot-ball.
How many more are playing foot-ball than base-ball?

There are 92 days in October, November, and December. In October there are 31 days, and in November 30.
How many days are there in December?

10. The process by which each of the preceding problems is solved is called **Subtraction.** The larger number in each problem is called the **Minuend**; the smaller, the **Subtrahend.** The answer in the first two problems is called the **Remainder**; in the last four, the **Difference.** Hence the following definitions:

A *Remainder* is the part of a number left after a part has been taken away.

A *Difference* is the number which must be added to a smaller number to produce a larger.

A *Minuend* is

1. The whole of a number from which a part is to be taken to find the remainder.
2. A larger number to be compared with a smaller to find the difference.

A *Subtrahend* is

1. The part of a number to be taken from the whole to find the remainder.
2. A smaller number to be compared with a larger to find the difference.

Subtraction is the process of finding a Difference or a Remainder.

NOTE I. The fact that two or more numbers are to be added may be indicated by writing the sign + between them. This sign, it will be observed, is an erect cross. It is called the PLUS SIGN, and may be read either "plus" or "and."

Thus, $6+4$ signifies that 6 and 4 are to be added, and may be read 6 plus 4, or 6 and 4; $8+10+11$ signifies that 8, 10, and 11 are to be added, and may be read 8 plus 10 plus 11, or 8 and 10 and 11.

The fact that a number is to be subtracted from another may be indicated by writing the subtrahend after the minuend with the sign — between them. This sign, which is a short horizontal line, is called the MINUS SIGN, and may be read either "minus" or "less."

Thus, $6-4$ signifies that 4 is to be subtracted from 6, and may be read 6 minus 4, or 6 less 4.

The fact that two arithmetical expressions are equal to each other may be indicated by writing the sign = between them. This sign, which consists of two short parallel horizontal lines, is called the SIGN OF EQUALITY, and may be read "equals" or "are."

Thus, $6+4=10$ signifies that the sum of 6 and 4 is equal to 10, and may be read 6 plus 4 equals 10, or 6 and 4 are 10.

NOTE 2. It is evident that the plus sign is understood between each two figures of a number. Thus 25 signifies 2 tens + 5 units, or 20 + 5; 675 signifies 6 hundreds + 7 tens + 5 units, or 600 + 70 + 5; 23.45 signifies 2 tens + 3 units + 4 tenths + 5 hundredths, or 20 + 3 + .4 + .05.

NOTE. 3. It is evident that only numbers of the same kind, or that are considered as being of the same kind, can be added. But the likeness may consist in only one condition or quality. Thus 2 boys + 3 girls = 5 children; 4 boys + 5 girls + 6 men + 7 women = 22 people; 6 chairs + 3 tables + 2 sofas = 11 pieces of furniture. But it is evident that what was really added was not 2 boys and 3 girls, but 2 children and 3 children; not 6 chairs, 3 tables, and 2 sofas, but 6 pieces of furniture, 3 pieces of furniture, and 2 pieces of furniture; not 4 boys, 5 girls, 6 men, and 7 women, but 4 people, 5 people, 6 people, and 7 people.

Numbers possessing a like quality are called, as regards that quality, LIKE NUMBERS. Numbers not possessing the common quality that is made the basis of comparison are called UNLIKE NUMBERS.

The preceding remarks concerning the addition of like numbers apply equally, of course, to their subtraction.

NOTE 4. It is also evident that when the element of magnitude is considered, like quantities must, if necessary, be reduced to the same magnitude before adding. Thus to add feet and inches, both must be reduced to feet or inches; to add hours and days, both must be reduced to hours or days; to add tenths and units, both must be reduced to tenths or units.

NOTE 5. To problems designed to develop a clear idea of a mathematical term or to logically establish a general mathematical principle the term INDUCTIVE is applied. A careful examination of the inductive exercises preceding Articles 9 and 10 will show the relation existing between addition and subtraction to be of the following nature:

In addition the parts are given and the whole required.

In subtraction the whole and all the parts but one are given, and that part is required.

11. It is presumed that the pupil has already mastered the first steps of addition and subtraction. If necessary, however, mentally complete the following table and practise upon it until you can read it without the least hesitation and with absolute accuracy.

Observe that in this table no addend, subtrahend, or remainder is greater than 9.

Table of Fundamental Combinations.

9 from 16 leaves	4 and 5 are	6 and are 14
3 and 5 are	6 from 10 leaves	9 from 12 leaves
4 and 7 are	3 and 3 are	9 and are 16
6 and 4 are	7 from 11 leaves	5 from 10 leaves
4 and 4 are	8 from 17 leaves	4 from 9 leaves
5 and 9 are	8 and are 17.	3 from 10 leaves
6 and 3 are	9 from 14 leaves	5 and are 10
3 and 9 are	7 and 4 are	6 and are 10
3 and are 10	6 from 9 leaves	5 and 3 are
7 from 10 leaves	7 and 7 are	6 and are 9
8 and 3 are	9 and are 12	5 from 12 leaves
6 and are 13	8 and 8 are	8 from 15 leaves
9 and 5 are	3 from 8 leaves	7 and 8 are
5 from 11 leaves	5 and are 12	6 and 7 are
9 from 18 leaves	4 and 6 are	9 and 8 are
7 and 3 are	8 from 14 leaves	7 from 12 leaves
8 from 11 leaves	9 and are 14	8 and are 15
5 from 8 leaves	9 and 9 are	8 and are 14
2 from 8 leaves	9 and 6 are	3 and are 8
7 and are 10	3 and 6 are	8 and 4 are
3 from 11 leaves	3 and are 9	4 and are 8
9 from 17 leaves	5 and are 8	9 and 8 are
8 from 16 leaves	5 from 13 leaves	9 and 7 are
4 from 13 leaves	9 and 3 are	8 and 6 are
9 and are 18	3 and 8 are	5 from 14 leaves
9 from 15 leaves	8 and 5 are	4 and 9 are
7 from 16 leaves	5 and are 14	9 and are 15
6 from 12 leaves	7 and 6 are	7 and are 11
9 and 8 are	9 from 13 leaves	4 and are 12
9 and are 17	5 and 6 are	6 from 15 leaves
3 and 7 are	6 from 14 leaves	5 and 8 are
4 from 8 leaves	5 and 4 are	7 and are 14
8 and 7 are	7 from 13 leaves	5 and 7 are
7 and 5 are	7 and are 16	8 from 13 leaves
9 and 4 are	8 and are 15	5 and are 11
4 from 11 leaves	3 from 12 leaves	9 and are 13
4 and are 13	4 from 10 leaves	6 and are 11
8 from 12 leaves	9 and are 15	6 and are 15
4 and are 10	7 and 9 are	8 and are 14
9 and 6 are	7 and are 12	8 and are 12

"13 is the sum of 4 and 9, of 5 and 8, and of 6 and 7." Name in the same way the different pairs of addends that produce each of the following amounts:

2 3 4 5 6 7 8 9 10 11 12 14 15 16 17 18

12. To Add when No Addend is Greater than 9.

To add a single column composed of more than two figures, we evidently may combine two of these numbers, then combine a third number with these two, and so proceed until all the numbers have been united. But observe the following combinations:

1. Treat numbers whose sum is 10 as one number

(1) If the numbers, whether two or more, are adjacent. Thus, in the first exercise, 7, 17, 26, 36.

(2) If the numbers are but two, and are separated by but a few figures. Thus, in the second exercise, 10, 15, 25, 35.

(1)	(2)	(3)	(4)	(5)
6	4	4	5	9
2	7	4	3	1
2	6	3	4	3
9	3	5	5	3
4	2	3	7	4
6	5	8	2	1
7	8	7	6	3
36	35	34	32	24

2. Treat two or more adjacent numbers as one number if their sum is 9 or less.

Thus, in the third exercise, 15, 23, 30, 34; in the fourth, 8, 15, 24, 32; and in the fifth, 8, 15, 24.

NOTE. No pupil aiming to become an expert accountant should be satisfied with his skill in addition until he possesses the ability to recognize the amount of every essential combination with as little mental effort as would be required to grasp the idea of a short and simple sentence. Thus, just as the mind receives as a whole the idea of the sentence "Shut the door," so whenever the combination $7+8$ is presented, the value 15 should suggest itself without hesitation or conscious effort.

13. To Prove Addition.

Does the sum of two or more numbers depend upon the order in which the numbers are added?

Give, then, a method for proving an addition.

If a series of numbers is divided into several groups, how will the result obtained by adding the sums of these groups compare with the sum of the series?

Give, then, a second method for proving an addition.

Ex. 15.

Observing the directions given in the preceding article, find the sum of each of the following columns:

1	2	3	4	5	6	7	8	9	10	11	12	13	14
3	3	7	7	1	3	6	5	7	7	9	3	5	9
4	3	4	6	0	2	7	2	2	8	5	6	6	9
5	1	2	0	0	5	5	7	8	7	5	0	9	9
8	4	8	9	6	7	3	8	1	5	7	5	7	4
6	5	5	2	6	0	4	6	2	1	5	7	5	7
2	2	7	3	3	1	9	3	1	6	8	0	9	3
4	8	1	0	6	4	1	4	4	2	2	1	7	5
4	1	6	7	2	2	6	1	4	4	1	1	4	8
6	6	2	6	5	2	5	7	1	3	3	3	8	1
2	8	1	6	4	4	2	7	4	1	8	5	3	3
9	8	6	4	8	6	7	1	2	3	6	2	5	4
7	6	8	3	5	2	7	1	1	4	4	6	3	1
8	6	8	3	3	9	6	7	4	8	3	5	2	4
0	5	2	6	4	4	5	3	7	8	2	5	2	7
5	8	8	9	6	5	2	7	1	6	6	8	9	5

15	16	17	18	19	20	21	22	23	24	25	26	27	28
9	6	5	7	2	8	3	6	4	6	8	0	8	3
8	7	8	9	6	8	7	4	5	6	4	4	6	5
5	6	7	3	6	7	2	9	9	9	8	7	7	7
8	6	9	7	8	3	7	7	7	7	7	2	1	2
1	5	6	3	4	7	5	2	5	3	2	9	4	3
4	5	2	4	4	8	1	3	8	4	7	6	2	8
2	9	2	8	4	2	2	5	5	4	8	2	7	1
2	7	4	7	3	2	2	9	3	6	5	3	2	2
5	3	5	2	8	6	4	3	2	7	4	3	5	8
1	3	6	7	4	2	3	1	6	9	3	6	3	1
7	4	5	7	2	1	8	9	5	4	3	7	5	3
7	4	6	7	2	8	4	6	6	6	8	4	5	2
8	4	6	2	1	7	8	6	8	2	2	4	6	7
7	5	4	3	2	1	6	8	7	9	9	4	4	3
2	5	2	7	4	7	4	3	5	1	6	4	1	1
5	8	3	6	6	2	5	6	2	3	3	8	1	2
7	4	3	4	6	7	1	3	8	5	6	4	8	5
7	2	8	4	4	6	7	8	2	5	4	4	2	6
7	8	4	5	7	3	5	3	2	8	7	8	8	5
7	3	7	4	5	1	1	1	4	4	5	3	2	8
4	4	8	2	6	3	6	2	1	2	2	5	8	7
6	8	4	3	2	4	6	8	5	3	6	8	6	7
6	5	4	3	2	3	2	8	6	4	5	3	8	7
1	8	1	6	3	3	4	5	4	7	4	8	2	3
6	3	7	4	7	2	5	3	6	7	8	3	5	4
3	7	5	6	3	4	8	6	3	2	5	5	4	6

14. Principles of Addition and Subtraction.

A certain man's property is in two investments. At the beginning of the year he estimates the income on each investment at a certain amount.

By what process can he find his probable total income?

The income from each investment forms what element of the operation?

The total income?

1. Suppose that the income from one investment is $500 greater than the expected income.

How will the total income compare with the expected total income?

Increasing an addend has, then, what effect on the amount?

2. Suppose that the income from one investment is $500 less than the expected income.

How will the total income compare with the expected total income?

Diminishing an addend has, then, what effect on the amount?

3. Suppose that the income from one investment is $500 greater, and from the other $500 less, than the expected income.

How will the total income compare with the expected total income?

Increasing one addend by a certain quantity and diminishing another by the same quantity has, then, what effect on the amount?

* * *

A man at the end of six months estimates his annual income at a certain amount, and his expenditures at a certain amount.

By what process can he find his probable net income?

The total income forms what element of the operation?

The expenditures?

The net income?

1. Suppose that the income is $500 greater than the expected income.

How will the net income compare with the expected net income?

Increasing the minuend has, then, what effect on the remainder?

2. Suppose that the income is $500 less than the expected income.

How will the net income compare with the expected net income?

Diminishing the minuend has, then, what effect on the remainder?

3. Suppose that the expenditures are $500 greater than the expected expenditures.

How will the net income compare with the expected net income?

Increasing the subtrahend has, then, what effect on the remainder?

4. Suppose that the expenditures are $500 less than the expected expenditures.

How will the net income compare with the expected net income?

Diminishing the subtrahend has, then, what effect on the remainder?

5. Suppose that the income is $500 greater than the expected income, and that the expenditures are $500 greater than the expected expenditures.

How will the net income compare with the expected net income?

Suppose that the income is $500 less than the expected income, and that the expenditures are $500 less than the expected expenditures.

How will the net income compare with the expected net income?

Increasing or diminishing both subtrahend and minuend by the same number has, then, what effect on the remainder?

For convenience of reference we here express the preceding principles.

PRINCIPLES OF ADDITION.

1. Increasing an addend increases the amount.

2. Diminishing an addend diminishes the amount.

3. Increasing one addend and diminishing another by the same number does not change the amount.

PRINCIPLES OF SUBTRACTION.

1. Increasing a minuend increases the remainder.

2. Diminishing a minuend diminishes the remainder.

3. Increasing a subtrahend diminishes the remainder.

4. Diminishing a subtrahend increases the remainder.

5. Increasing or diminishing both minuend and subtrahend by the same number does not change the remainder.

15. To Prove Subtraction.

What relation has a minuend to a subtrahend and a remainder?

Give, then, a method of proving a subtraction.

16. To Add when One, or More, of the Addends is Greater than 9.

We are to find the sum of 876, 934, 568, 827, 913, 475, 822, and 936.

What is the sum of the units' column?

41 units is equal to how many units and how many tens?

What do we do with the 1 unit?

What with the four tens?

What is the sum of the tens' column?

35 tens is equal to how many tens and how many hundreds?

What do we do with the 5 tens?

What with the 3 hundreds?

What is the sum of the hundreds' column?

What, then, is the total sum?

3 4
876
934
568
827
913
475
822
936
6351

Find the sum of 143.56, 897.16, 479.92, 723.42, 974.33.

EXPLANATION.

We first write the numbers, so that units will stand under units, tenths under tenths, etc. We next add the right-hand column. Its sum we find to be .19, or 1 tenth and 9 hundredths. The 9 hundredths we write under the hundredths' column; the 1 tenth we add, or "carry," to the tenths' column.

The sum of the tenths' column, including the 1 tenth brought from the hundredths' column, is 2.3, or 2 units and 3 tenths. The 3 tenths we write under the tenths' column; the 2 units we carry to units' column. Proceeding in the same way with the other columns, we have as the result of the addition, 3218.39

SOLUTION.

3 2 2 1
143.56
897.16
479.92
723.42
974.33
3218.39

NOTE 1. In finding the sums in the preceding exercises the mental processes are as follows:

1. 8, 16, 23, 31, 41; 11, 20, 30, 35; 11, 20, 25, 33, 42, 46, 54, 63.
2. 5, 13, 19; 6, 16, 23; 5, 12, 21, 28; 6, 15, 24, 31; 4, 12, 16, 23, 32.

NOTE 2. It will be observed that in the preceding exercises the left-hand figure of the sum of each column is written above the column containing the figures of its order. The object of this is to make clear the process of 'carrying.' In practice, the recording of the left-hand figure is advisable only when the columns to be added are of unusual length.

NOTE 3. In adding very long columns, the necessity of burdening the mind with the number of hundreds may be avoided by placing a light mark near the last figure of each hundred.

Ex. 17.

Add the following numbers:

(1)	(2)	(3)	(4)	(5)	(6)	(7)
29	209	1649	20979	466350	9908765	66543279
91	916	6089	77799	663377	7909963	99777554
13	851	1931	82173	237436	7276443	42242633
32	434	2111	74111	711263	2474327	46362636
77	176	7924	96593	238221	2194195	12971898
43	247	5177	61927	169435	1765742	42775285
48	174	4397	77476	458682	7884186	86672728
88	235	5589	24644	845271	4329562	95714469
32	164	4566	44036	536161	6612553	54144510
54	515	4642	15153	546615	2564445	64065114
40	521	4230	20155	402563	2314224	55066334
12	431	2161	60300	432224	8632266	64630416
22	425	4243	20062	153653	4016513	40754022
27	252	1041	10251	731366	1342347	11055461
34	102	1100	33568	401122	1532000	14655402
12	312	8031	12422	323314	2300151	74350208

(8)	(9)	(10)	(11)	(12)	(13)	(14)
25	789	6432	36900	533877	9087954	43799589
46	603	8431	18053	532068	2074226	53053162
58	875	8857	76865	604853	4548673	21324354
75	567	6040	28542	346875	1054486	35684621
99	557	4389	90000	542143	9468750	68421578
55	660	3467	12254	876543	4267745	99990000
33	228	6574	38954	163240	5678900	86421576
76	864	6876	48391	234500	2817463	81475589
78	608	3652	45728	195680	7352422	92735400
92	715	8957	20904	505527	5759674	90906040
21	435	4075	16416	579554	2385140	77258829
87	799	4134	12555	026414	5448456	61562853
35	621	7800	17515	117611	4911121	75315602
18	111	2111	26663	053443	1021672	18870000
74	621	1612	55443	183406	2183121	21481812
12	200	1070	61250	405110	3044250	77002253
30	205	2034	64345	150230	1001663	10000317
48	413	4784	57837	583718	7214855	17517518
75	537	7735	73583	555824	7288824	82197179
67	469	4757	36869	974938	5746213	59378555
65	923	4845	58285	348987	8281554	87653422
49	716	3921	76354	171195	9558958	65656968
97	658	7767	58655	955665	5859556	69987985
99	277	4523	78221	272172	3237317	77111878
18	384	2186	56984	139700	3689024	67329909
56	728	1956	82340	638754	1470975	46112990

Ex. 18.

Add the following numbers:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
214	772	609	621	521	963	265	9478	3249
290	457	219	900	243	998	876	1453	8900
445	878	215	906	800	510	995	3214	7631
211	945	756	908	517	875	342	1334	3167
427	580	100	809	625	900	527	8020	4100
578	964	480	752	640	200	955	9780	8007
421	578	604	790	680	400	804	4009	4950
807	800	608	900	800	920	760	7620	7008
689	295	700	500	405	866	867	6000	9669
356	589	116	809	124	572	215	6421	9862
147	742	654	873	956	212	553	7447	8113
324	890	346	667	175	938	366	9771	7194
932	926	447	695	721	544	478	2952	4246
032	161	532	966	517	497	384	7777	2768
513	834	192	259	489	271	618	1474	2246
828	336	221	642	343	399	626	8346	7983
783	531	592	370	399	636	223	7942	2494

(10)	(11)	(12)	(13)
123,462,403	374,874,488	167,236,111	587,671,249
465,247,434	633,225,440	289,314,178	552,164,314
228,876,171	723,702,543	270,583,611	210,704,051
121,581,172	748,014,814	971,873,183	603,214,251
682,112,663	767,476,650	960,345,292	417,688,241
400,934,669	212,179,162	888,666,241	184,767,930
957,780,168	418,049,350	928,154,611	186,646,416
468,584,532	682,631,176	261,082,012	543,863,766
815,652,715	287,461,661	402,583,263	449,956,015
950,366,648	622,646,779	232,641,828	843,212,648
589,001,123	238,788,784	960,417,348	188,241,887

(14)	(15)	(16)	(17)
722,197,111	421,019,891	703,101,251	932,394,409
956,065,404	404,244,244	955,850,047	230,219,104
600,830,093	334,013,026	209,133,058	511,493,540
641,135,901	241,814,528	779,698,531	482,818,903
384,799,143	417,449,823	658,614,732	121,241,037
326,422,799	333,434,344	438,733,743	373,075,228
904,140,801	568,111,627	705,597,454	252,452,665
556,915,522	881,352,809	415,059,423	110,100,790
200,101,003	720,000,790	926,792,901	766,711,881
915,699,754	225,372,388	944,564,561	472,376,753
537,675,588	211,857,638	902,467,679	552,370,331
633,376,590	274,767,872	991,271,774	494,453,291
304,270,057	995,481,545	655,555,572	727,932,993

Ex. 19.

Add the following numbers:

(1)	(2)	(3)	(4)
718404.69	4297258.59	250471.209	533986.48
270542.50	7025652.68	607780.665	110890.11
855120.36	1322196.39	704674.423	906095.80
999496.69	6644496.46	623272.771	9273791.21
911297.33	5174723.39	337787.637	3584207.41
21854.13	5263596.61	659007.869	422739.35
6309.33	77305.68	426776.508	27198.48
802944.46	2910177.75	65588.960	910766.06
913420.54	3129112.15	511595.110	322101.11
558703.46	9502002.61	683772.478	169494.48
53608.13	203225.85	274712.385	389104.66
783724.12	8495992.58	70042.938	532477.87
341305.49	316369.80	323298.252	2790159.21
132200.65	191003.40	27894.281	878846.13
27322.32	1057486.70	109465.457	6682243.09
21731.44	114300.10	122500.360	9036903.18
233440.19	114300.10	122500.360	6682243.09
2173.10	135270.10	427714.138	1044205.64
198725.13	40280.19	21030.424	263750.31
599208.40	11595.11	184980.338	853301.37
53685.97	4912970.47	325683.940	116319.90
10959.29	1649903.28	15094.500	21083.41
622700.65	2359787.16	164755.181	868067.74
332286.94	8643094.21	346991.872	276631.85
44943.65	155118.58	27190.255	46810.35
65002.30	258200.16	97613.005	21231.52
252.26	1832057.50	485928.320	542123.34
112036.38	4567890.22	683514.090	6712.35
496854.15	6185823.76	806592.144	643575.20
397236.80	9045174.13	487433.446	480966.74
719468.24	7145297.54	957421.643	536827.78
911816.26	1013653.86	133691.558	235010.33
639329.05	8060408.70	120700.525	203941.41
150155.15	8100103.57	502602.979	319025.14
412158.11	3556597.88	698760.759	615521.21
207610.01	6383023.02	932029.740	753015.03
354121.11	4551079.33	748857.504	875354.02

17. Before attempting to find the sum in each of the following exercises, arrange the numbers to be added so that units of the same order will fall in the same vertical column.

Ex. 20.

Add the following numbers:

1. 947,347,856.45; 92,036.008; 453.934,7; 57,328; 964,832.754; 327.834,96.

2. 3,845.95; 978.070,8; 76,829.345; 7,006.934; 582.796,8; 1,324.85; 319,587.946.

3. 74,166.42; 89,991.924; 1,796.041,3; 46,871.049; 3,586.24.

4. 1,796.2; 3,911.000,08; 69,422.001; 211,664.835,12; 24,410.67; 444,256.25.

5. 2,256,884.1; 371,235.265; 54,476.482,6; 7,123,157,38; 48,629.777,5; 248.841; 94.673,771.

6. 3,000,126.7; 47,924.500,1; 497,300,000.000,2; 479.624; 2,567,864.05; 1,432,699.006; 38,452,840.1.

7. 1,045,061.1; 325.42; 956,427.854; 68,539.431; 2,000,000.903; 145,769.74; 9,000,101.000,472.

8. 161,893.17; 87,456.6; 255.329,46; 857,437.954,2; 89,742,226,432.104; 25,354.100,07.

9. 5,835,205.20; 600,320.6; 50,208.700; 320,679.50; 7,786,825.009; 25.02; 6,328.65; 845,936.020.

10. 45,675.25; 9,236.030; 4,289.25; 2,658.723; 3,258.63; 241.720; 7,933.940.

11. 43,250.963; 6,050.02; 689.003; 9.02; 560.28; 33,675.3; 4,589.300.

12. 67,933.250; 7,458.678; 75,876.98; 53,985.07; 6,783.25; 795.05; 55.67; 3,678.95.

13. 5,679.456; 542.06; 32.458; 9,458.965; 523.76; 1,865.020; 468.95.

14. 935,463.002; 46,893.01; 365,840.250; 46,270.30; 968,530.235; 75,963.24; 540.685.

15. 25,940.25; 6,586.04; 75,900.45; 43,001.50; 6,948.06; 40,630.78; 7,985.95.

16. 83,250.45; 9,285.36; 80,520.99; 4,368.25; 28,395.48; 44,325.789; 75,982.75.

Mental Addition.

One may have occasion to add mentally numbers of several columns. To do this, first find the sum of the left-hand column. Reduce this sum to units of the next lower order, combine with it the second column, and so proceed until all the columns have been added.

Explanations.

1. We are to add 350, 392, 478, 555, 197, 324, 832.

Commencing with the left-hand column and adding, we have 11, 21, 27, 2-70 (two-seventy); 2-75, 2-84, 2-91, 2-96, 3-05, 3-10, 31-00 (thirty-one zero zero); 31-06, 31-13, 31-18, 31-28.

2. We are to add 7056, 8922, 1090, 2840, 3725, 3384, 5603.

Commencing with the left-hand column and adding, we have 8, 13, 22, 29, 2-90; 2-99, 3-07, 3-14, 3-23, 32-30; 32-40, 32-44, 32-53, 32-60, 326-00; 326-08, 326-17, 326-20.

Exercises.

(1)	(2)
350	7056
392	8922
478	1090
555	2840
197	3725
324	3384
832	5603
3128	32620

Ex. 21.

Add the following numbers:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
225	325	199	957	475	265	543	875	826
436	856	786	754	297	512	634	648	734
987	497	564	801	851	716	532	721	807

(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)
789	945	286	569	785	412	465	367	216
462	286	475	796	543	217	680	275	623
764	342	129	437	695	423	789	523	565
532	734	432	825	511	487	678	233	436

(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)	(27)
826	846	539	432	125	687	412	523	188
435	789	397	321	286	264	567	356	247
764	943	409	944	324	793	358	579	418
937	215	457	354	457	465	847	276	601
246	793	140	510	312	465	676	387	452
327	100	298	297	452	375	564	477	481

(28)	(29)	(30)	(31)	(32)	(33)	(34)	(35)
4321	1901	4372	5255	7689	7040	6325	4672
9276	7643	5691	6546	4318	8900	6437	8931
3472	2107	1099	2776	5656	8216	9024	3406

18. To Subtract when the Subtrahend or the Remainder is Greater than 9.

We are to subtract 4863 from 6749.	6749
3 units from 9 units leave how many units ?	4863
Can we subtract 6 tens from 4 tens ?	1886

Suppose that we add ten tens to the minuend.
How many hundreds must we add to the subtrahend to counterbalance this addition ?
6 tens from 14 tens leave how many tens ?

How many hundreds have we to subtract from the 7 hundreds ?
Can we subtract 9 hundreds from 7 hundreds ?
Suppose that we add ten hundreds to the minuend ?
How many thousands must we add to the subtrahend to counterbalance this addition ?
9 hundreds from 17 hundreds leave how many hundreds ?

How many thousands have we to subtract from the 6 thousands ?
5 thousands from 6 thousands leave how many thousands ?

4863 subtracted from 6749 leaves, then, what remainder ?

		SOLUTIONS.		
		(1)	(2)	(3)
Ex. 1.	Subtract 2364 from 6876.	6876	87.48	6849
Ex. 2.	Subtract 32.92 from 87.48.	2364	32.92	4963
Ex. 3.	Subtract 4963 from 6849.	4512	54.56	1886

EXPLANATIONS.

Ex. 1. We are to subtract 2364 from 6876.

For convenience we so write the subtrahend that each of its orders will fall under the corresponding order of the minuend.

4 units from 6 units leave 2 units, 6 tens from 7 tens leave 1 ten 3 hundreds from 8 hundreds leave 5 hundreds, and 2 thousands from 6 thousands leave 4 thousands. Therefore, 2364 subtracted from 6876 leaves 4 thousands, 5 hundreds, 1 ten, and 2 units, or 4512.

Ex. 2. We are to subtract 32.92 from 87.48.

2 hundredths from 8 hundredths leave 6 hundredths. We cannot subtract 9 tenths from 4 tenths. We therefore add 10 tenths to the minuend. The effect of this addition upon the remainder we must counterbalance by adding 1 unit to the subtrahend.

9 tenths from 14 tenths leave 5 tenths, 3 units from 7 units leave 4 units, and 3 tens from 8 tens leave 5 tens. Therefore, 32.92 subtracted from 87.47 leaves 54.56.

Ex. 3. Explain the solution of Ex. 3.

Ex. 22.

Find the remainders in the following exercises:

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
967	865	696	755	894	589	697	548	987
653	721	435	523	413	322	631	424	563

(10)	(11)	(12)	(13)	(14)	(15)	(16)
43154	76439	97689	57435	99678	54786	42758
22464	68357	78496	25673	89784	45697	31686
(17)	(18)	(19)	(20)	(21)	(22)	(23)
65834	88574	97873	89754	98785	42878	78979
24523	75352	85432	35423	73253	31153	55321

(24)	(25)	(26)	(27)	(28)
5678947	7583256	3289768	9583218	8676325
4869554	5632498	3195879	8976428	6895449

(29)	(30)	(31)	(32)	(33)	(34)	(35)
76325	98765	45832	83294	95542	85932	43285
49324	89624	28491	76845	84561	76543	32198
(36)	(37)	(38)	(39)	(40)	(41)	(42)
59812	21345	78156	85432	72123	61614	83468
43822	12357	19567	68543	64587	25765	49369
(43)	(44)	(45)	(46)	(47)	(48)	(49)
23145	96321	46873	63214	87632	38543	88775
14648	84732	39584	54327	15785	25674	79999
(50)	(51)	(52)	(53)	(54)	(55)	(56)
95421	82131	75312	32134	52175	83467	75134
37265	72473	28757	13849	45732	25357	59867
(57)	(58)	(59)	(60)	(61)	(62)	(63)
69845	32565	24958	48572	72315	59723	63245
48978	28754	19879	35983	54987	48874	57352
(64)	(65)	(66)	(67)	(68)	(69)	(70)
75632	85634	96732	83256	59632	89763	73245
59749	76956	49864	74689	48948	45874	68496
(71)	(72)	(73)	(74)	(75)	(76)	(77)
12324	43256	56873	63258	76533	16542	63258
6595	35687	47984	49876	67645	9876	35985

Ex. 23.

Subtract

1. 684248 from 962428
2. 423 from 6580.
3. 5347 from 7206.
4. 5974 from 63258.
5. 42897 from 351462.
6. 52879 from 838000.
7. 386 from 7953.
8. 4975 from 648923.
9. 7548 from 953225.
10. 84257 from 952753.
11. 259763 from 752253.
12. 37952 from 72834.
13. 583 from 74325.
14. 4285 from 62153.
15. 538976 from 820013.
16. 38656 from 89742.

Ex. 24.

Find the difference between

1. 8974327 and 5906824.
2. 84325 and 63540.
3. 97438 and 6475.
4. 863529 and 7978.
5. 7428936 and 934268.
6. 8364285 and 5789098.
7. 938425 and 650047.
8. 648932 and 9354.
9. 6328576 and 5865976.
10. 94482 and 86735.
11. 895325 and 75328.
12. 854846 and 753269.
13. 994623 and 837865.
14. 953418 and 869745.
15. 6351750 and 49786.
16. 387592 and 8974.

Ex. 25.

1. From 9304206 there is taken away 327804. What is the remainder?

2. From 856262 there is taken away 76984. What is the remainder?

3. From 64231 there is taken away 5487. What is the remainder?

4. From 8534261 there is taken away 7986465. What is the remainder?

5. From 42764 there is taken away 3685. What is the remainder?

Ex. 26.

1. The larger of two numbers is 567804 and the smaller 56284. What is the difference?

2. The larger of two numbers is 75532 and the smaller 28967. What is the difference?

3. The larger of two numbers is 831244 and the smaller 79986. What is the difference?

4. The larger of two numbers is 7524125 and the smaller 6889. What is the difference?

5. The larger of two numbers is 94675 and the smaller 87659. What is the difference?

Mental Exercises in Addition and Subtraction.

19. The fundamental processes of addition and subtraction have been described under articles 13 and 14. One should, however, accustom himself to solve mentally such examples as are given under the following Exercises:

NOTE 1. The pupil should practise upon combinations similar to those given in Ex. 27, until he can name results with a single mental effort.

NOTE 2. The examples under Ex. 28 can evidently be solved by an application of the first and the second principles of addition, and those under Ex. 29 by a similar application of the first and the second principles of subtractiou. Thus, to add 99 to 537, we add 100 and subtract 1 from the amount; to add 96 to 725, we add 100 and subtract 4 from the amount; and to add 104 to 233 we add 100 and add 4 to the amount. Following a similar principle, to subtract 99 from 875 we subtract 100 and add 1 to the remainder; to subtract 96 from 307 we subtract 100 and add 4 to the remainder; and to subtract 107 from 430 we subtract 100 and subtract 7 from the remainder.

Name the results in the following exercises:

Ex. 27.

12 + 12	25 + 9	15 + 17	87 − 6	24 − 12
12 + 11	74 + 7	14 + 13	93 − 8	23 − 11
10 + 12	96 + 9	12 + 15	45 − 9	21 − 10
11 + 11	324 + 12	15 + 17	64 − 7	21 − 11
10 + 11	875 + 9	16 + 14	782 − 12	22 − 10

Ex. 28.

537 + 99	3255 + 999	846 + 101	725 + 96
949 + 99	9764 + 999	432 + 102	892 + 98
824 + 99	8227 + 999	793 + 103	347 + 97
347 + 99	4936 + 999	233 + 104	213 + 95
559 + 99	5640 + 999	839 + 105	5860 + 998
682 + 99	3224 + 999	752 + 106	7614 + 988
733 + 99	4160 + 999	387 + 107	6723 + 995

Ex. 29.

875 − 99	8457 − 999	982 − 101	835 − 96
438 − 99	5204 − 999	736 − 102	748 − 97
215 − 99	7648 − 999	305 − 103	432 − 97
764 − 99	4250 − 999	524 − 104	307 − 96
990 − 99	6459 − 999	736 − 105	4370 − 998
543 − 99	2327 − 999	827 − 106	8304 − 988
112 − 99	1628 − 999	430 − 107	4294 − 995

Ex. 30.

The following exercises will form a general review of notation, addition, and subtraction.

1. Find the sum of nine hundred thirty-four billion one hundred five thousand seven, and one hundred three thousandths four ten-thousandths; eight trillion two hundred five billion sixteen thousand, and seven hundred millionths; and two billion thirty-six million five hundred forty thousand, and two hundred eight thousandths nine hundred sixteen billionths seven ten-billionths.

2. Subtract eight trillion five million four, and five billionths six ten-billionths from three hundred nine trillion seven hundred eighty-four billion two hundred eighty-nine, and two hundred four millionths twenty-four hundred-millionths.

3. Find the sum of nine hundred forty-six trillion eight hundred thirty-four million sixteen, and eight hundred millionths thirty-three billionths seventy-six hundred-billionths; four hundred eight billion, and two thousandths seven hundred billionths; eighty-two trillion one, and four ten-trillionths; three hundred twenty-nine thousand eight, and two hundred forty thousandths fifteen millionths five billionths five hundred-billionths; and forty-six million nine hundred eighty-three, and two hundred seven millionths six trillionths thirty-seven hundred-trillionths.

4. Subtract forty-eight billion one hundred thousand nine hundred thirty-four, and five hundred sixty-seven thousandths three hundred eight millionths forty-six billionths five hundred four trillionths from ninety-six billion five thousand four, and fourteen millionths seven ten-billionths.

5. Find the sum of two hundred four trillion nine million eleven, and fourteen thousandths five millionths seven hundred-billionths; eight billion four hundred million four thousand six, and nine hundred thousandths nine hundred-thousandths; seven hundred trillion nine hundred seven billion nine hundred forty-nine million one hundred, and five thousandths two hundred millionths sixty-five hundred-billionths.

6. Subtract three hundred four billion seven million eight hundred eleven, and seven millionths four hundred billionths from nine hundred billion eleven million eleven.

20. Problems in Addition and Subtraction.

In an arithmetical exercise, as the term is more commonly used, one is required simply to perform a certain operation. In a problem it is necessary first to determine by a course of reasoning what operation must be performed. The character of this reasoning in certain classes of problems is indicated by the following explanations:

Prob. 1. The difference between two numbers is 2874. The smaller number is 9346. What is the larger number?

Prob. 2. The difference between two numbers is 937. The larger number is 4256. What is the smaller number?

Prob. 3. A merchant sells during the year $125,000 worth of goods. He receives from groceries $49,375.47, and from hardware $11,804.93. His remaining receipts are from dry-goods. What does he receive from dry-goods?

SOLUTIONS.

(1)	(2)	(3)
2874	4256	49375.47
9346	937	11804.93
12220	3319	61180.40
		125000.00
		63819.60

(4)	(5)
89894951	1835700825
66585947	2770217344
23309004	4605918169

Prob. 4. Suppose that Venus and the Earth are on the same side of the Sun and in direct line with it. When so situated what is the least possible distance between them? (See note and diagram after Exercise 12.)

Prob. 5. Suppose that Uranus and Neptune are on opposite sides of the Sun and in direct line with it. When so situated what is the greatest possible distance between them?

Explanations.

Prob. 1. By definition a difference is a number which must be added to a smaller number to produce a larger. Therefore, the larger number in the problem is the sum of 2874 and 9346, or 12220.

Prob. 2. From the definition of a difference it follows that a difference and a subtrahend equal a minuend. Therefore, a minuend may be thought of as a whole, and the subtrahend and a number equal to the difference as its two parts. It follows, therefore, that a given part, 937, must be subtracted from the given whole, 4256. $4256 - 937 = 3319$. The answer to the problem, therefore, is 3319.

Prob. 3. The amount received from groceries and from hardware must be the sum of $49,375.47 and $11,804.93, or $61,180.40. The receipts from dry-goods must be the difference between $125,000 and $61,180.40, or $63,819.60.

Prob. 4, As Venus and the Earth are on the same side of the Sun, the distance between them must be the difference of their solar distances.

The distance between them is to be the least possible distance. The less a minuend the less the remainder, and the greater a subtrahend the less the remainder. We therefore use as a minuend 89,894,951, the minimum solar distance of the earth, and as a subtrahend 66,585, 947, the maximum solar distance of Venus. 89, 894,951 — 66,585,947 = 23,309,004. The distance between the two planets, therefore, is 23,309,004 miles.

Prob. 5. As Uranus and Neptune are on opposite sides of the Sun, the distance between them must be the sum of their solar distances.

The distance between them is to be the greatest possible distance. The greater an addend the greater the amount. We therefoɹe use as addends, 1,835,700,825 the maximum distance of Uranus, and 2, 770,217,344, the maximum distance of Neptune. 1,835,700,825 + 2,770,217, 344 = 4,605,918,169. The distance between the two planets, therefore, is 4,605,918,169 miles.

6. A farmer feeds to his stock in a certain year 1135 bushels of corn. He raises 789 bushels. How many bushels must he buy?

7. There is in the treasury of a certain town at the beginning of the year $8769.79. During the next six months there is added $1595.47, and there is taken out $8324.52. At the end of the six months how much is there in the treasury?

8. The remainder in a certain exercise is 5347. The subtrahend is 8406. What is the minuend?

9. The remainder in a certain exercise is 834. The minuend is 2369. What is the subtrahend?

10. A father leaves one of his sons $7800, another $9700, and a third $2540 less than the amount left to the first two. How much does he leave to the three?

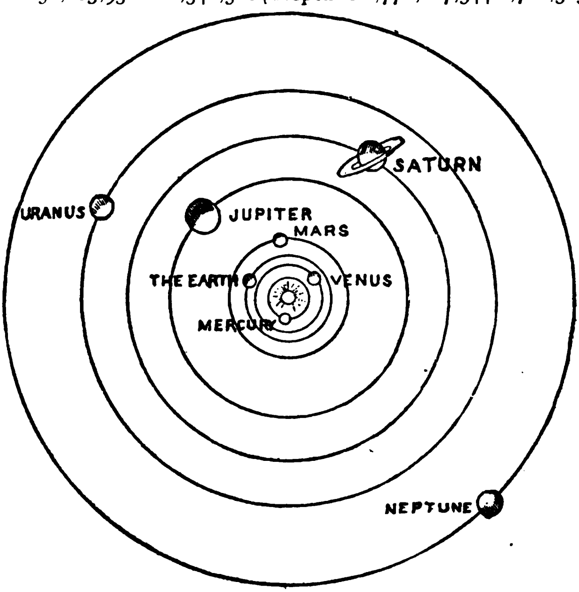
11. A farmer sells three horses. For the first he receives $175, for the second $225, and for the third $125. He buys

eight cows and pays for them $160 less than he received for the horses. What does he pay for the cows?

12. In 1890 the population of Greater New York was 2,492,591; of Philadelphia, 1,046,964; of Chicago, 1,099,850; of St. Louis, 451,770; of Boston, 448,477; of Baltimore, 434,439. In 1900 the population of Greater New York was 3,437,202; of Philadelphia, 1,293,697; of Chicago, 1,698,575; St. Louis, 575,238; of Boston, 560,892; of Baltimore, 508,957. What was the increase in the population of each city and of the six cities from 1890 to 1900.

NOTE. The following table shows the names of the different planets and their maximum and minimum distances from the sun, while the accompanying diagram illustrates their position relative to the sun and to each other. After a careful study of the table and the diagram the pupil should experience no difficulty in solving the problems that are based upon them.

Mercury	42,665,560	28,119,716	Jupiter	498,603,768	452,782,530
Venus	66,585,947	65,677,009	Saturn	921,105,027	823,164,139
Earth	92,965,489	89,894,651	Uranus	1,835,700,825	1,672,001,279
Mars	152,283,936	126,340,516	Neptune	2,770,217,344	2,722,325,120



13. Suppose that Venus and the Earth are on the same side of the Sun and in direct line with it. When so situated what is the greatest possible distance between them?

14. Suppose that Venus and the Earth are on opposite sides of the Sun, and in direct line with it. When so situated what is the least possible distance between them?

15. Suppose that the two planets are situated as stated in Problem 14. When so situated what is the greatest possible distance between them?

16. Suppose that Uranus and Neptune are situated on the same side of the Sun and in direct line with it. When so situated what is the least possible distance between them?

17. Suppose that Saturn and Neptune are on opposite sides of the Sun and in direct line with it. When so situated what is the greatest possible distance between them?

18. Suppose that the two planets are situated as stated in Problem 17. When so situated what is the least possible distance between them?

19. Suppose that the two planets are situated as stated in Problem 16. When so situated what is the greatest possible distance between them?

20. Find the maximum and the minimum distance between Mars and the Earth when on opposite sides of the Sun and in direct line with it.

21. Find the maximum and the minimum distance between Mars and the Earth when on the same side of the Sun and in direct line with it.

22. Find the maximum distance between Mercury and Venus when on opposite sides of the Sun and in direct line with it.

23. Find the minimum distance between Mercury and Venus when on the same side as the Sun and in direct line with it.

24. Assume all the planets to be at their maximum distance, on the same side of the Sun, and in direct line with it. When so situated what is the distance between the Earth and each of the other planets?

REVIEW QUESTIONS.

Define addition; an addend; a sum or amount.

Define a remainder; a difference; a minuend; a subtrahend; subtraction.

Read the expression $8 + 10$ and explain its signification. Define the plus sign.

Read the expression $6 - 4$ and explain its signification. Define the minus sign.

Read the expression $6 + 4 = 10$ and explain its signification Define the sign of equality.

Define like numbers; unlike numbers. Are 6 chairs, 3 tables, and 2 sofas like or unlike numbers ?

Explain the addition of feet and inches; of hours and days; of tens and units.

Define inductive problems. By means of inductive problems what relation is shown to exist between addition and subtraction?

What is the first step in addition and subtraction?

Complete and recite the table under Art. 11.

How is a single column composed of more than two figures added ? Explain the addition of $7 + 6 + 4$; of $8 + 5 + 2 + 3 + 6 + 7 + 4$; of $8 + 5 + 2 + 3 + 6$. Give the note under Art. 12.

Give two methods for proving addition.

Give the first principle of addition; the second; the third.

Give the first principle of subtraction; the second; the third; the fourth; the fifth.

Explain the addition of 143.56, 897.16, 479.92, 723.42, and 974.33. Give Note 2, Art. 16; give Note 3.

Add mentally 826, 435, 764, 937, 246, 327. Explain the process.

Explain the subtraction of 32.92 from 87.48.

Give two methods for proving subtraction.

* * *

Add 327 and 672. Explain the process. Give a rule for adding numbers of more than one column when the sum of no column is greater than 9.

Add 372 and 672. Explain the process. Give a rule for adding numbers of more than one column when one or more of the sums is greater than 9.

Add mentally 569, 796, 437, 825. Explain the process. Give a rule for adding mentally numbers of more than one column.

Subtract 2364 from 6876. Explain the process. Give a rule for subtracting when the subtrahend or the remainder is greater than 9.

Multiplication and Division.

1. A farmer has a cart that will hold 35 bushels of grain and a bin that will hold 4 cart-loads. Find by two distinct methods how many bushels the bin will hold.

35	35
35	4
35	140
35	
140	

2. He has a second bin that is 5 times as large as the first. Find by two distinct methods how many bushels the second bin will hold.

3. He has a third bin that will hold 3 times as much as the first two combined. How many bushels will the third bin hold?

21. The longer process of solving these problems, as we have previously learned, is called Addition. The shorter process is called **Multiplication.** When problems are solved by this shorter process special names are given to the numbers to be united and to the final results. Thus, 35 bushels, the amount of grain in each load in the first problem, and the corresponding element in each of the other problems, is called, not an Addend, but a **Multiplicand.** In like manner, 140 bushels of grain, the result in the first problem, and the corresponding result in each of the other problems, is called, not the Sum or Amount, but the **Product.** It is evident, moreover, that the 4 in the first problem, the 5 in the second, and the 3 in the third are elements that do not appear in the ordinary process of addition. These elements, which show how many equal numbers are to be combined, are called **Multipliers.** Hence the following definitions:

Multiplication is a shortened process of combining two or more equal numbers.

The *Multiplicand* is one of the equal numbers to be thus combined.

The *Multiplier* is the number which shows how many equal numbers are to be thus combined.

The *Product* is the result of the shortened process of combination.

NOTE. Observe that the second part of the third problem can be solved either by addition or by multiplication, but that the first part can be solved only by addition.

A farmer has a bin that will hold 140 bushels of grain. He has a cart that will hold 35 bushels. How many loads will it take to fill the bin?

$$\begin{array}{rl} 35 & (1) \\ \underline{35} & (2) \\ 70 & \\ \underline{35} & (3) \\ 105 & \\ \underline{35} & (4) \\ 140 & \end{array} \qquad \begin{array}{r|l} 140 & \underline{35} \\ 140 & \underline{4} \end{array}$$

22. The shorter process of solving the preceding problem is called **Division.** Considering, therefore, simply the nature of the process, division may be defined as follows:

Division is a shortened process of finding how many given equal numbers must be combined to produce another given number.

NOTE. Addition, Subtraction, Multiplication, and Division are commonly referred to as the four fundamental processes. It is evident, however, that the process of addition is really the foundation of the three remaining processes. Thus, if one is acquainted with all the combinations in addition through $9+9$ he can determine, for example,

(1) What number must be added to 8 to produce 17.

(2) The sum of any number of 8's.

(3) The number of 8's that must be combined to produce any given number.

Definitions based upon the preceding definition might be given of all the terms used in division. It will be found especially profitable, however, to frame and fix in mind definitions based upon the character of the problems in which the process of division is employed. These problems may be divided into the following classes:

1. It may be required to find how many equal parts must be added to produce a given whole.

Thus in the preceding problem we find that 35 bushels must be added 4 times to produce 140 bushels.

2. It may be required to find how many given equal smaller numbers must be added to produce a number equal to a given larger number.

Thus, we may suppose that the farmer has a second bin that will hold 35 bushels, and we may wish to ascertain the relation of the quantity in the first bin to the quantity in the second bin. In solving this problem we find that 35 bushels must be added 4 times to produce a quantity of grain equal to the quantity in the larger bin.

3. It may be required to find the magnitude of that equal part which, added a given number of times, will produce a given whole.

Thus, we may suppose that the farmer sells one-fourth of the grain in the first bin, and we may wish to ascertain the number of bushels that he sells. In solving this problem we find that 35 bushels is the number of bushels that, added 4 times, will produce 140 bushels.

4. It may be required to find the magnitude of that smaller number which, added a given number of times, will produce a number equal to a given larger number.

Thus, we may suppose that the farmer has a second bin one-fourth as large as the first, and we may wish to ascertain the number of bushels in the second bin. In solving this problem we find that 35 bushels is the number of bushels that, added 4 times, will produce a quantity of grain equal to the 140 bushels in the larger bin.

5. It may be required to find what relation a given part bears to a given whole.

Thus, we may suppose that the farmer sells from the larger bin 35 bushels, and we may wish to find the simplest expression for the relation of the number of bushels that he sells to the total number of bushels. In solving this problem, following principles that will be explained later, we find that 35 bushels bear the same relation to 140 bushels that 1 bears to 4, and that the simplest expression to indicate this relation is $\frac{1}{4}$.

6. It may be required to find what relation a given smaller number bears to a given larger number.

Thus, we may wish to find the relation of the capacity of the smaller bin to the capacity of the larger bin. In solving this problem we find, as in the preceding problem, that 35 bushels bear the same relation to 140 bushels that 1 bears to 4, and that the simplest expression to indicate this relation is $\frac{1}{4}$.

23. The number of bushels in the bin in the first problem, and the corresponding element in each of the other problems, is called a **Dividend**; the number of bushels in the cart in the first problem, and the corresponding element in each of the other problems, is called a **Divisor**; and the result in each problem is called a **Quotient.**

Hence the following definitions:

A *Dividend* is

1. A number to be compared with, or measured by, another given number.

2. A number to be separated into a given number of equal parts.

A *Divisor* is

1. A number with which to compare, or measure, another given number.

2. A number showing into how many equal parts a given number is to be separated.

A *Quotient* is the result of a division. It may show

1. How many times a divisor must be added to produce a dividend.

2. The relation of the magnitude of the dividend to the magnitude of the divisor.

While there are several classes of problems in division, all problems are solved by the same process. Therefore, practically, every quotient shows how many times the divisor must be added to produce the dividend. Therefore, the divisor multiplied by the quotient must equal the dividend. Hence, the dividend as related to the divisor and the quotient is a product, while the divisor and the quotient as related to the dividend are factors. Division, therefore, with reference to the class of problems it is used to solve, may be defined as

"The process of finding one of the two factors when the product and the other factor are given."

NOTE 1. As previously explained numbers are commonly used in connection with quantities, and when so used are called Concrete Numbers. Thus the basis of the first problem under multiplication is 35 bushels of grain. A number may, however, simply indicate

(1) How many times a number is to be added.

(2) The relation in magnitude which one number bears to another,

Thus, in the first problem in Multiplication, we are to add the concrete number 35 bushels 4 times; in the first problem in Division we are to find how many times we must add the concrete number 35 bushels to produce the concrete number 140 bushels; and in the fifth problem in Division we are to find the number that expresses the relation that the concrete number 35 bushel bears to the concrete number 140 bushels. Numbers thus used are called **ABSTRACT NUMBERS**.

The term "abstract number" is also applied to numbers that are used without reference to any unit whatever. Thus in finding the sum of 10, 15, and 25 we refer to these numbers as abstract numbers. It is, however, self-evident that an actual problem cannot exist in which the numbers involved do not refer to definite units.

NOTE 2. As the multiplicand and the product correspond respectively to the numbers added and the amount, it is evident (see Note 3. Art. 10) that they must always be like numbers.

NOTE 3. The multiplier and the multiplicand, in distinction from the product, are called **FACTORS**. Thus 4 and 35 are factors of 140, 5 and 27 are factors of 135, and 9 and 2 are factors of 18.

NOTE 4. The product obtained by taking a number two or more times as a factor is called its **POWER**. Thus 4, which is the product of two multiplied by 2, and 8, which is the product of 2 multiplied by 2 multiplied by 2, are powers of 2; and 9, which is the product of 3 multiplied by 3, and 27, which is the product of 3 multiplied by 3 multiplied by 3, are powers of 3.

The second power of a number is commonly called its **SQUARE**, and the third power its **CUBE**. Thus, 4 is called the square, and 8 the cube of 2; 9 is called the square, and 27 the cube of 3. The appropriateness of these special terms will be explained under Mensuration.

NOTE 5. When a factor is to be multiplied by itself, or "raised to a power," the number of times it is to appear is indicated by writing a small figure at the right and above it. Thus the fact that 3 is to be taken as a factor 4 times is indicated by the expression 3^4. The number thus written to show how many times the factor is to be taken is called its **EXPONENT**. Thus, in the preceding expression 4 is the exponent of 3.

NOTE 6. When a power is given and its equal factors are required, one of these factors is called a **ROOT**. Thus 5 is the Square Root of 25, 4 is the Cube Root of 64, and 2 is the Fifth Root of 32.

NOTE 7. When a number is to be separated into equal factors, the operation may be indicated by placing before the number a sign called the **RADICAL SIGN**.

If the second, or square, root is to be taken, the radical sign alone is employed. If a higher root than the square is to be taken, the number of equal factors into which the number is to be separated, or the **DEGREE** of the root, is indicated by placing a small figure called the **INDEX** at the left of and above the radical sign. Thus, the expression $\sqrt{25}$ indicates the square root of 25; and the expression $\sqrt[3]{27}$, the cube root of 27.

NOTE 8. The sign of multiplication is an inclined cross placed between the factors. It is read "times" or "multiplied by." Thus, 8×7 is read "8 times 7," or "8 multiplied by 7," and $9\times 25\times 48$ is read "9 times 25 times 48," or "9 multiplied by 25 multiplied by 48."

NOTE 9. The complete sign of division is a combination of a horizontal line and a colon, written between the dividend and the divisor. Thus, $56\div 8$ is read "56 divided by 8."

NOTE 10. The quotient, when showing the relation in magnitude which the dividend has to the divisor, is commonly called the **RATIO**. Thus, instead of asking, "What is the quotient of 240 bushels divided by 35 bushels?", the more concise expression is, "What is the ratio of 140 bushels to 35 bushels?"

In indicating a ratio the horizontal line of the complete sign of division is commonly omitted. Thus the ratio of 140 to 35 is indicated by the expression $140:35$.

The term ratio is used in a second sense as a name for the expression of relation. Thus, "In the ratio (expression) $140:35$ the ratio (quotient) is 4."

NOTE 11. An expression indicating that two ratios are equal to each other is called a **PROPORTION**. Thus, $5:10=18:36$, which signifies that the ratio of 5 to 10 equals the ratio of 18 to 36, is called a proportion. The more common sign of proportion is the double colon. Thus the preceding proportion may be written $5:10::18:36$ and read 5 is to 10 as 18 is to 36.

NOTE 12. When a dividend is less than a divisor, the division may be indicated by writing the dividend above and the divisor below the horizontal line of the complete sign of division. Thus the division of 3 by 4 may be indicated by the expression $\frac{3}{4}$.

NOTE 13. The expression $\frac{3}{4}$ and all other expressions of division, whether the dividend or the divisor is the larger, in which the dividend is written above the divisor, are called FRACTIONS. A fraction is read by reading first its dividend, and then its divisor with "th" or "ths" added as the divisor is 1 or greater than 1. Thus, $\frac{3}{4}$ is read three-fourths.

In printed matter a fraction may be expressed by writing the dividend before and the divisor after the horizontal line. Thus, five-sixteenths may be written 5-16.

The formation of the name of the fraction is irregular if the denominator is 2 or 3. Thus, $\frac{1}{2}$ is read one-half; $\frac{3}{2}$ is read three-halves; and $\frac{2}{3}$ is read two-thirds.

If the name of the divisor ends in "ve," "th" is added and ve is changed to f. Thus $\frac{1}{5}$ is read one-fifth, $\frac{1}{12}$ is read one-twelfth.

If the name of the divisor ends in "y," "eth" is added and y is changed to i. Thus $\frac{1}{20}$ is read one-twentieth.

A fraction is commonly defined as "One or more of the equal parts of a unit." When thus considered the divisor of the fraction, from the fact that it shows the name, or denomination, of the parts into which the unit is divided, is called a DENOMINATOR, and the dividend, as showing the number of parts, is called a NUMERATOR. Thus 2 is the numerator and 3 the denominator of the fraction $\frac{2}{3}$.

The numerator and denominator of a fraction are called its TERMS. Thus 2 and 3 are the terms of the fraction $\frac{2}{3}$.

In this text-book all operations in fractions will be based upon the idea that a fraction is an indicated expression of division. The pupil, therefore, should thoroughly master the fundamental principles of division, which will be developed later, as having done this his work in fractions will be but the intelligent application of previously acquired ideas.

24. Products where Neither Factor is Greater than 12.

The first step in multiplication and division is to memorize the following table, which shows all products where neither factor is greater than 12, and all quotients where neither divisor nor quotient is greater than 12.

NOTE. The table should be read in the following manner:

Twice 2 are 4, 2 in 4 twice. 2 times 3 are 6, 3 times 2 are 6; 3 in 6 twice, two in 6 three times

Twice 6 are 12, 6 times 2 are 12; 6 in 12, twice, 2 in 12, 6 times. 3 times 4 are 12, 4 times 3 are 12; 4 in 12, 3 times, 3 in 12, 4 times.

Multiplication and Division Table.

4 = 2×2	28 = 4×7	60 = 6×10
6 = 2×3	30 = 3×10	63 = 7×9
8 = 2×4	5×6	64 = 8×8
9 = 3×3	32 = 4×8	66 = 6×11
10 = 2×5	33 = 3×11	70 = 7×10
12 = 2×6	35 = 5×7	72 = 8×9
3×4	36 = 4×9	6×12
14 = 2×7	3×12	77 = 7×11
15 = 3×5	6×6	80 = 8×10
16 = 2×8	40 = 4×10	81 = 9×9
4×4	5×8	84 = 7×12
18 = 2×9	42 = 6×7	88 = 8×11
3×6	44 = 4×11	90 = 9×10
20 = 2×10	45 = 5×9	96 = 8×12
4×5	48 = 4×12	99 = 9×11
21 = 3×7	6×8	100 = 10×10
22 = 2×11	49 = 7×7	108 = 9×12
24 = 2×12	50 = 5×10	110 = 10×11
3×8	54 = 6×9	120 = 10×12
4×6	55 = 5×11	121 = 11×11
25 = 5×5	56 = 7×8	132 = 11×12
27 = 3×9	60 = 5×12	144 = 12×12

The pupil should so thoroughly memorize the above facts as to be able to give without an instant's hesitation the missing element in each of the following oral exercises.

18 is times 6	21 is times 7	42 is times 7
is 3 times 12	is 4 times 6	is 7 times 8
20 is times 10	is 3 times 5	99 is 9 times
is 3 times 6	is 2 times 11	54 is times 9
36 is 3 times	is 2 times 7	108 is times 12
28 is times 7	12 is times 2	80 is times 10
is 3 times 10	is 2 times 12	is 10 times 11
12 is 2 times	24 is times 12	is 5 times 8
27 is times 9	60 is 6 times	90 is times 10
24 is 4 times	96 is 8 times	77 is 7 times
30 is times 10	48 is 6 times	40 is 5 times
is 3 times 3	is 7 times 10	55 is times 11
is 8 times 11	45 is 5 times	is 10 times 12
is 7 times 7	66 is 6 times	42 is times 7

is 5 times 9	144 is 12 times	is 10 times 10
77 is times 11	80 is 8 times	84 is times 12
60 is times 12	66 is times 11	110 is 10 times
72 is 8 times	90 is 9 times	is 5 times 11
42 is 6 times	is 4 times 12	is 7 times 8
is 9 times 10	88 is 8 times	55 is 5 times
is 7 times 11	is 11 times 11	12 is 12 times
45 is times 9	is 7 times 12	is 6 times 10
132 is 11 times	96 is times 12	60 is times 10
63 is 7 times	132 is times 12	84 is 7 times
is 4 times 11	44 is 4 times	120 is times 12
is 11 times 12	is 8 times 10	44 is times 11
60 is 5 times	120 is 10 times	56 is times 8
is 9 times 12	48 is times 8	88 is times 11
40 is times 8	is 9 times 11	is 6 times 9
64 is 8 times	121 is 11 times	is 6 times 7
70 is 7 times	56 is 7 times	88 is times 11
72 is times 12	63 is times 9	is 5 times 12

Ex. 31.

Separate each of the following numbers into pairs of factors. Let no factor be greater than 12.

Thus, the factors of 28 are 4 and 7; the factors of 36 are 3 and 12, and 4 and 9, and 6 and 6.

4	12	20	27	35	45	55	66	81	99	121
6	14	21	28	36	48	56	70	84	100	132
8	15	22	30	40	49	60	72	88	108	144
9	16	24	32	42	50	63	77	90	110	
10	18	25	33	44	54	64	80	96	120	

25. Composite Numbers and Prime Numbers.

A number that can be separated into factors is called a **Composite Number,** and a number that cannot be separated into factors a **Prime Number.** Hence, a factor that cannot itself be separated into factors is called a **Prime Factor.**

NOTE. By observing the following directions one can at a glance separate into prime factors any of the numbers given in the preceding list:

1. Memorize the prime factors of 8 and 12.

2. To separate into factors numbers like 54, 48, 120, etc., that have more than two factors, proceed as follows:

(1) First separate the numbers into two factors.

(2) Separate each of these factors into its prime factors.

Thus, the factors of 54 are 6 and 9, or 2 and 3, and 3 and 3; the factors of 48 are 6 and 8, or 2 and 3, and 2, 2 and 2; and the factors of 120 are 10 and 12, or 2 and 5, and 2, 2 and 3.

Ex. 32.

Separate into their prime factors each of the numbers under Ex. 31.

26. In case the dividend is not divisible by the divisor, two courses may be followed.

1. The part left may be written at the right of the quotient and separated from it by a dash. Thus, the result obtained by dividing 27 by 4 may be written 6—3. When so written the part left is called the **Remainder.**

2. It may be written above the divisor and placed at the right of the quotient. When so written it forms a fraction and is considered a part of the quotient. Thus, the result obtained by dividing 27 by 4 may also be written 6¾.

Ex. 33.

Name the quotient and the remainder in each of the following exercises:

54÷7	68÷7	82÷8	96÷11	108÷12
55÷7	119÷12	130÷11	84÷11	139÷12
85÷9	111÷11	58÷6	41÷6	42÷9
98÷11	140÷11	99÷12	43÷9	56÷12
57÷8	110÷12	131÷12	86÷6	59÷7
70÷8	71÷9	120÷11	121÷11	127÷11
80÷9	94÷11	105÷12	118÷12	51÷6
66÷8	128÷11	137÷12	40÷9	55+7
58÷7	73÷7	68÷7	74÷8	84÷9
37÷7	47÷8	45÷6	73÷7	89÷9
34÷8	53÷9	79÷8	79÷9	28÷5
55÷9	23÷3	37÷4	59÷5	23÷4
68÷9	73÷9	50÷7	22÷5	60÷7
26÷9	42÷5	63÷8	42÷9	36÷8
82÷7	83÷9	62÷9	23÷8	49÷5
62÷6	93÷9	37÷9	57÷5	43÷9

26. To Multiply when the Multiplicand is Greater than 12.

A teacher taught three terms of school of 12 weeks each. The first term she received $7 a week, the second term $8, and the third $9.

What was her salary for the first term?
For the second term?
For the third term?
How can we find her salary for the three terms?
How, then, when our multiplicand consists of several parts can we find the total product?

A study of the preceding processes will show us how to multiply when the multiplicand is so large that it must be separated into several parts. We evidently must multiply each of the several parts by the common multiplier, and unite the several products into one total product.

We are to multiply 1157 by 8.	1157
What sign is understood between each two figures of the multiplicand?	8
Of how many parts, then, does the multiplicand consist?	9256

8 times 7 units are how many units?
56 units are how many units and how many tens?
What do we do with the 6 units?
What shall we do with the 5 tens?

8 times 5 tens are how many tens?
What do we do with the 5 tens of the first product?
40 tens plus 5 tens are how many tens?
45 tens are how many tens and how many hundreds?
What do we do with the 5 tens?
What shall we do with the 4 hundreds?

8 times 1 hundred are how many hundreds?
What do we do with the 4 hundreds of the second product?
8 hundreds plus 4 hundreds are how many hundreds?
12 hundreds are how many hundreds and how many thousands?
What do we do with the 2 hundreds?
What shall we do with the 1 thousand?

8 times one thousand are how many thousands?
What do we do with the 1 thousand of the third product?
8 thousands plus 1 thousand are how many thousands?
What, then, is the total product of 1157 and 8?
Give, then, a rule for multiplying when the multiplicand is greater than 12 and the multiplier is not greater.

NOTE. It is evident that each of the products by 8 might have been written entire. In that case the final step of the solution would have been to unite the several products into one total product.

	SOLUTIONS.		
	(1)	(2)	(3)
Ex. 1. Multiply 323 by 3.	969	26505	96108
Ex. 2. Multiply 2945 by 9.			
Ex. 3. Multply 8009 by 12.			

EXPLANATIONS.

Ex 1. We are to multiply 323 by 3.

3 times 3 units are 9 units,

3 times 2 tens are 6 tens. That the 6 tens may occupy their proper position in the product we write them at the left of the 9 units.

3 times 3 hundreds are 9 hundreds. That the 9 hundreds may occupy their proper positions in the product we write them at the left of the 6 tens. We thus obtain as our total prodnct 969.

Ex. 2. We are to multiply 2945 by 9.

9 times 5 units are 45 units, or 5 units and 4 tens. We write down the 5 units and reserve the 4 tens for the next partial product.

9 times 4 tens are 36 tens. 36 tens plus the 4 tens reserved from the first product are 40 tens, or 0 tens and 4 hundreds. We write the 0 tens at the left of the 5 units, and reserve the 4 hundreds for the next partial product.

9 times 9 hundreds are 81 hundreds. 81 hundreds plus the 4 hundreds reserved from the second product are 85 hundreds, or 5 hundreds and 8 thousands. We write the 5 hundreds at the left of the 0 tens, and reserve the 8 thousands for the next partial product.

9 times 2 thousands are 18 thousands. 18 thousands plus the 8 thousands reserved from the third product are 26 thousands. We write the 26 thousands at the left of the 5 hundreds, and have as a final product 26505.

Ex. 3. We are to multiply 8009 by 12.

12 times 9 uuits are 108 units. As there are neither tens nor hundreds in the multiplicand, we write down 108 as the three right-hand figures of the product.

12 times 8 thousands are 96 thousands. Writing the 96 thousands at the left of the 1 hundred, we have as a final product 96108.

Mental Processes in Preceding Exercises—3, 6, 9 .45; 36, 40; 81 85; 18, 26. 108, 96.

NOTE. It is customary before performing a multiplication to write the multiplier beneath the multiplicand and to draw a horizontal line to separate the multiplier from the position to be occupied by the product. This preliminary work, however, is unnecessary, as the multiplier, when not greater than 12, can easily be kept in mind while the multiplication is being performed.

Ex. 34.

1. $42353 \times 2 = ?$
2. $32642 \times 3 = ?$
3. $58263 \times 5 = ?$
4. $92431 \times 6 = ?$
5. $54255 \times 8 = ?$
6. $32489 \times 7 = ?$
7. $63248 \times 2 = ?$
8. $89433 \times 9 = ?$
9. $34562 \times 12 = ?$
10. $42344 \times 12 = ?$
11. $65347 \times 11 = ?$
12. $82432 \times 9 = ?$
13. $83423 \times 8 = ?$
14. $33457 \times 5 = ?$
15. $25634 \times 3 = ?$
16. $34816 \times 7 = ?$
17. $26348 \times 12 = ?$
18. $34563 \times 9 = ?$
19. $42333 \times 11 = ?$
20. $65782 \times 7 = ?$
21. $93475 \times 4 = ?$
22. $33463 \times 3 = ?$
23. $45638 \times 12 = ?$
24. $95634 \times 8 = ?$

Ex. 35.

Multiply

1. 432 by 2
2. 934 by 7
3. 526 by 4
4. 324 by 9
5. 436 by 7
6. 227 by 5
7. 763 by 6
8. 932 by 8
9. 215 by 2
10. 198 by 7

Multiply

11. 6042 by 6
12. 1894 by 5
13. 2936 by 4
14. 3227 by 8
15. 4326 by 3
16. 9807 by 5
17. 6405 by 9
18. 5726 by 6
19. 6987 by 7
20. 4324 by 2

Multiply

21. 64035 by 9
22. 62302 by 2
23. 74385 by 6
24. 23459 by 8
25. 98076 by 4
26. 47032 by 5
27. 59706 by 7
28. 93854 by 5
29. 50465 by 3
30. 94385 by 2

Ex. 36.

Find the product of

1. 934327 and 6
2. 242541 and 7
3. 243767 and 8
4. 764126 and 6
5. 673178 and 3
6. 525618 and 6
7. 456343 and 8
8. 342758 and 7
9. 637382 and 6
10. 735836 and 5
11. 342629 and 4
12. 772731 and 9
13. 414342 and 7
14. 146343 and 6
15. 848431 and 6
16. 852064 and 8
17. 472018 and 4
18. 743329 and 9
19. 317553 and 5
20. 172531 and 3
21. 831367 and 5
22. 956442 and 5
23. 642782 and 7
24. 824576 and 3
25. 776215 and 6
26. 624743 and 6
27. 695522 and 8
28. 787842 and 2
29. 732489 and 4
30. 484743 and 7
31. 230855 and 4
32. 513513 and 5
33. 553218 and 3
34. 146317 and 6
35. 831902 and 5
36. 831355 and 5
37. 842435 and 9
38. 575385 and 7
39. 658536 and 6
40. 672826 and 5
41. 466373 and 8
42. 987244 and 5
43. 273423 and 5
44. 482462 and 8
45. 236872 and 2

27. To Divide when the Quotient is to be Greater than 12.

A farmer has three bins of grain. The first contains 20 bushels, the second 23 bushels, and the third 26 bushels. In drawing the grain to market he puts it in barrels that hold three bushels each.

How many barrels can be filled from the first bin?
How many bushels will be left?
What may be done with these two bushels?

How many bushels will there then be in the second bin?
How many barrels can be filled from it?
How many bushels will be left?

What may be done with this one bushel?
How many bushels will there then be in the third bin?
How many barrels can be filled from it?

We have found that 6 barrels can be filled from the first bin; that 8 barrels can be filled from the second bin combined with the amount remaining in the first bin; and that 9 barrels can be filled from the third bin combined with the amount remaining in the second bin. How can we find the number of barrels that can be filled from the three bins?

How, then, when our dividend consists of several parts can we find the total quotient?

A study of the preceding processes will show us how to divide when the dividend is so large that it must be separated into several parts. We evidently must divide each of the several parts by the common divisor, and unite the several quotients into one total quotient.

$$3\overline{)2352}$$
$$784$$

We are to divide 2352 by 3.

What sign is understood between each two figures of the dividend?

Of how many parts, then, does the dividend consist?

3 is contained in 23 how many times?
What do we do with the 7?
What with 2, the remainder?
What, then, is our second partial dividend?

3 is contained in 25 how many times?

How does the order of the second partial dividend compare with the order of the first?

How, then, will the order of the second quotient figure compare with the order of the first?

Where, then, shall we write 8, the second quotient figure?
What shall we do with 1, the second remainder?
What, then, is our third partial dividend?

3 is contained in 12 how many times?

How does the order of the third partial dividend compare with the order of the second?

How, then, will the order of the third quotient figure compare with the order of the second?

Where, then, shall we write 4, the third quotient figure?

7 8 4 represents what number?

What, then, is the complete quotient obtained by dividing 2352 by 4?

Give, then, a rule for dividing when the quotient is to be greater than 12.

NOTE. Compare the first and the second of the preceding inductive exercises, and observe the advantage derived from the decimal system of notation in the following respects:

1. With reference to combining each remainder with the next lower order of the dividend.

2. With reference to combining the several partial quotients.

		SOLUTIONS.		
		(1)	(2)	(3)
Ex. 1.	Divide 969 by 3.	969	26509	96108
Ex. 2.	Divide 26509 by 9.	323	$2945\frac{4}{9}$	8009
Ex. 3.	Divide 96108 by 12.			

EXPLANATIONS.

Ex. 1. We are to divide 969 by 3.

3 is contained in 9 3 times, 3 in 6 twice, and 3 in 9 3 times.

Each partial dividend is one order lower than the preceding partial dividend. Each quotient figure, therefore, is one order lower than the preceding quotient figure, and the complete quotient is 323.

Ex. 2. We are to divide 26509 by 9.

9 is contained in 26 twice, with a remainder 8. The 8 we reduce to the next lower order and combine with 5, the third figure of the dividend. We thus obtain 85 as the second partial dividend.

9 is contained in 85 9 times, with a remainder 4. The second partial dividend is one order lower than the first, therefore the second quotient figure is one order lower than the first, and we place the 9 at the right of the 2.

The fourth figure of the dividend is 0. For the third partial dividend, therefore, we simply reduce the remainder 4 to the next lower order. We thus obtain 40 as the third partial dividend.

9 is contained in 40 4 times with a remainder 4. The third dividend is one order lower than the second, etc.

Complete the explanation. Explain the solution of Ex. 3.

Ex. 37.

1. $349 \div 3 = ?$
2. $659 \div 2 = ?$
3. $640 \div 5 = ?$
4. $496 \div 6 = ?$
5. $348 \div 9 = ?$
6. $876 \div 4 = ?$
7. $343 \div 9 = ?$
8. $789 \div 8 = ?$
9. $189 \div 2 = ?$
10. $732 \div 5 = ?$
11. $533 \div 3 = ?$
12. $321 \div 5 = ?$
13. $794 \div 8 = ?$
14. $4026 \div 5 = ?$
15. $9843 \div 7 = ?$
16. $6343 \div 9 = ?$
17. $5404 \div 2 = ?$
18. $8920 \div 5 = ?$
19. $3406 \div 3 = ?$
20. $4389 \div 5 = ?$
21. $6500 \div 4 = ?$
22. $6329 \div 8 = ?$
23. $4938 \div 5 = ?$
24. $1254 \div 5 = ?$
25. $3508 \div 7 = ?$
26. $4306 \div 9 = ?$
27. $54323 \div 7 = ?$
28. $84206 \div 3 = ?$
29. $34206 \div 2 = ?$
30. $56473 \div 3 = ?$
31. $34254 \div 6 = ?$
32. $98533 \div 5 = ?$
33. $65424 \div 6 = ?$
34. $63258 \div 3 = ?$
35. $73264 \div 5 = ?$
36. $24365 \div 2 = ?$
37. $56245 \div 3 = ?$
38. $64789 \div 5 = ?$
39. $80425 \div 7 = ?$

Ex. 38.

What is

1. $\frac{1}{2}$ of 542624
2. $\frac{1}{3}$ of 784234
3. $\frac{1}{3}$ of 983450
4. $\frac{1}{4}$ of 313660
5. $\frac{1}{8}$ of 524164
6. $\frac{1}{8}$ of 567210
7. $\frac{1}{4}$ of 531901
8. $\frac{1}{4}$ of 751641
9. $\frac{1}{2}$ of 614574
10. $\frac{1}{8}$ of 754210
11. $\frac{1}{3}$ of 256371
12. $\frac{1}{4}$ of 296672
13. $\frac{1}{3}$ of 342951
14. $\frac{1}{2}$ of 641680
15. $\frac{1}{4}$ of 913267
16. $\frac{1}{4}$ of 515743
17. $\frac{1}{8}$ of 616486
18. $\frac{1}{2}$ of 721746
19. $\frac{1}{3}$ of 631538
20. $\frac{1}{2}$ of 551752
21. $\frac{1}{3}$ of 343788
22. $\frac{1}{2}$ of 623489
23. $\frac{1}{8}$ of 747132
24. $\frac{1}{4}$ of 271704
25. $\frac{1}{2}$ of 621764
26. $\frac{1}{8}$ of 615243
27. $\frac{1}{2}$ of 379017
28. $\frac{1}{8}$ of 457810
29. $\frac{1}{4}$ of 356179
30. $\frac{1}{4}$ of 742790

Ex. 39.

1. $35632 \div 7 = ?$
2. $63498 \div 8 = ?$
3. $84734 \div 5 = ?$
4. $98746 \div 5 = ?$
5. $64517 \div 7 = ?$
6. $45726 \div 6 = ?$
7. $72198 \div 8 = ?$
8. $54634 \div 6 = ?$
9. $61423 \div 8 = ?$
10. $41538 \div 5 = ?$
11. $23549 \div 8 = ?$
12. $63325 \div 7 = ?$
13. $75782 \div 5 = ?$
14. $98380 \div 6 = ?$
15. $45733 \div 3 = ?$
16. $99522 \div 6 = ?$
17. $45175 \div 8 = ?$
18. $75830 \div 5 = ?$
19. $51745 \div 2 = ?$
20. $20062 \div 5 = ?$
21. $77742 \div 9 = ?$
22. $74768 \div 7 = ?$
23. $74213 \div 4 = ?$
24. $83153 \div 9 = ?$
25. $45632 \div 7 = ?$
26. $66441 \div 5 = ?$
27. $98234 \div 4 = ?$
28. $45652 \div 5 = ?$
29. $82177 \div 5 = ?$
30. $65418 \div 6 = ?$
31. $80458 \div 7 = ?$
32. $53267 \div 7 = ?$
33. $71146 \div 2 = ?$
34. $74280 \div 8 = ?$
35. $56215 \div 6 = ?$
36. $73357 \div 7 = ?$

28. To Multiply when the Multiplier is Greater than 12.

A carriage dealer has a stock of carriages that he sells at $70 each. The first week in June he sells 3 of these carriages, the second week 6, the third week 5, and the fourth week 7.

How many dollars' worth does he sell the first week?

How many dollars worth the second week?

The third week?

The fourth week?

How can we find the number of dollars' worth that he sells in the four weeks?

How, then, can we multiply when our multiplier consists of several parts?

We are to multiply 3486 by 7035.	3486
What sign is understood between each two figures of the multiplier?	7035
Of how many parts, then, does it consist?	17430
	10458
3486 × 5 gives what product?	24402
3486 × 3 gives what product?	24524010

How does the order of the 3 compare with the order of the 5?

How, then, will the order of the product by the 3 compare with the order of the product by the 5?

Where, then, must the right-hand figure of the second product be placed with reference to the right-hand figure of the first product?

3486 × 7 gives what product?

How does the order of the 7 compare with the order of the 3?

How, then, will the order of the product by the 7 compare with the order of the product by the 3?

Where, then, must the right-hand figure of the third product be placed with reference to the right-hand figure of the second product?

We have found the product of 3846 by each of the parts of the multiplier. How shall we find the prodnct by the entire multiplier?

How, then, do we multiply when our multiplier consists of several figures?

Ex. 1. Multiply 856 by 47.

Ex. 2. Multiply 2864 by 16.

Ex. 3. Multiply 5283 by 8009.

SOLUTIIONS.

(1)	(2)	(3)
5992	2864	47547
3424	17184	42264
40232	45824	42311547

EXPLANATIONS.

We are to multiply 856 by 47.

We first multiply 856 by 7. The product thus obtained is 5992.

We next multiply 856 by 4. 4 times 856 are 3424. Our second

partial product, therefore, is 3424. But the 4 of the multiplier is one order higher than the 7. It therefore follows that the second product is one order higher than the first, and that the right-hand figure of the second product must be written one order to the left of the right-hand figure of the first product.

To find the total product we combine our partial products. We thus find the total product to be 40232.

Ex. 2. We are to multiply 2864 by 16.

We first multiply 2864 by 1. Once 2864 is 2864. Our first partial product, therefore, is 2864.

We next multiply 2864 by 6. 6 times 2864 are 17184. Our second partial product, therefore, is 17184.

But the 6 of the multiplier is one order lower than the 1. It therefore follows that the second product is one order lower than the first, and that the right-hand figure of the second product must be written one order to the right of the right-hand figure of the first product.

To find the total product we combine our two partial products. We thus find the total product to be 45824.

Ex. 3. We are to multiply 5283 by 8009.

We first multiply 5283 by 9. 9 times 5283 are 47547. Our first partial product, therefore, is 47547.

We next multiply 5283 by 8. 8 times 5283 are 42264. Our second partial product, therefore, is 42264. But the 8 of the multiplier is 3 orders higher than the 9. It follows, therefore, that the second product is three orders higher than the first, and that the right-hand figure of the second product must be written three places to the left of the right-hand figure of the first product.

To find the total product we combine our two partial products. We thus find the total product to be 42311547.

NOTE. To multiply by numbers like 819 multiply first by the 1, and then by either the 8 or the 9. Why?

Ex. 40.

Multiply

1. 3446 by 352	6. 64955 by 5342	11. 784356 by 32526
2. 5237 by 604	7. 62988 by 6359	12. 554379 by 42077
3. 6009 by 459	8. 45592 by 5062	13. 809490 by 45393
4. 4758 by 609	9. 74258 by 6079	14. 809225 by 73509
5. 5208 by 365	10. 56789 by 9758	15. 865325 by 70043

Ex. 41.

Multiply

1. 85302 by 9922
2. 83113 by 8005
3. 64213 by 1099
4. 92452 by 1124
5. 15433 by 7433
6. 31153 by 3197
7. 11007 by 5191
8. 13445 by 3755
9. 19085 by 6006
10. 53308 by 1277
11. 90549 by 5559
12. 11911 by 1642
13. 42657 by 1095
14. 42116 by 1435
15. 41185 by 7415
16. 15367 by 8541
17. 83017 by 9108
18. 33301 by 4418
19. 55184 by 5508
20. 15375 by 5043
21. 53414 by 4719
22. 55331 by 7401
23. 14278 by 5531
24. 43327 by 4102

Ex. 42.

Multiply

1. 270064 by 50604
2. 536079 by 30009
3. 800723 by 10908
4. 743189 by 94209
5. 531115 by 74193
6. 600422 by 60075
7. 865006 by 52128
8. 500708 by 55002
9. 852704 by 50021
10. 501605 by 90051
11. 980558 by 50032
12. 364809 by 83309
13. 742485 by 69005
14. 559052 by 96606
15. 754227 by 10008
16. 100964 by 55809
17. 411906 by 62518
18. 432893 by 55708
19. 742989 by 80986
20. 754227 by 52001
21. 642009 by 80099
22. 270421 by 75081
23. 553908 by 85802
24. 640932 by 50164
25. 655905 by 90053
26. 200098 by 90315
27. 558909 by 54409
28. 425004 by 42098
29. 200988 by 68002
30. 330098 by 73306

Ex. 43.

Find the product of

1. 54246 × 1342
2. 24653 × 2115
3. 34866 × 2653
4. 17899 × 1349
5. 74566 × 1299
6. 85123 × 2358
7. 78597 × 8012
8. 27479 × 1765
9. 20095 × 2759
10. 24879 × 9123
11. 35315 × 4136
12. 24896 × 2134
13. 14522 × 2675
14. 34692 × 7001
15. 12399 × 7789
16. 24095 × 2354
17. 24875 × 1788
18. 70095 × 7325
19. 77009 × 1223
20. 91229 × 6012
21. 22441 × 3212
22. 34756 × 9765
23. 24487 × 7852
24. 12609 × 9923
25. 23495 × 6885
26. 17841 × 7522
27. 13486 × 2555
28. 23229 × 5667
29. 75588 × 9912
30. 99556 × 1784

29. To Divide when the Divisor is Greater than 12.

5272272	5904
47232	893
54907	
53136	
17712	
17712	

We are to divide 5272272 by 5904.

How many figures at the left of the dividend must be taken to obtain a partial dividend divisible by 5904?

What, then, is our first partial dividend?

5904 is nearly how many thousands?

6 thousands are contained in 52 thousands how many times?

5904, then, is contained at least how many times in 52722?

How do we ascertain whether it is contained only 8 times?

5904×8 gives what product?

Suppose that this product had been greater than the partial dividend. What conclusion should we have arrived at concerning the magnitude of the first quotient figure?

$52722 - 47232 = ?$

Suppose that this remainder had been greater than the divisor. What conclusion should we have arrived at concerning the quotient figure?

Explain the formation of the second partial dividend.

6 thousands are contained in 54 thousands how many times?

5904 must, then, be contained in 54907 at least how many times?

How would the error be shown if 9 were two large a quotient figure?

If it were too small a quotient figure?

Explain the formation of the third partial dividend.

6 thousands are contained in 17 thousands nearly how many times?

5904, then, may be contained in 17712 how many times?

Do we find it to be contained 3 times?

What, then, is our total qnotient?

Give, then, a rule for dividing when the divisor is greater than 12.

Ex. 1. Divide 40232 by 856.

Ex. 2. Divide 45824 by 2864.

Ex. 3. Divide 42311547 by 5283.

SOLUTIONS.

(1)

40232	856
3424	47
5992	
5992	

(2)

45824	2864
2864	16
17184	
17184	

(3)

42311547	5283
42264	800
47547	
47547	

We are to divide 40232 by 856.

We first find how many figures at the left of the dividend must be used to secure a partial dividend that will contain the divisor.

8, the first figure of the divisor, is larger than 4, the first figure of the dividend. It is therefore evident that the first partial dividend must contain one more figure than the divisor, and that this dividend will be 4023.

8 hundreds are contained in 40 hundreds exactly 5 times. 856, therefore, cannot be contained in 4023 more than 4 times.

$856 \times 4 = 3424$, a number less than 4023, and $4023 - 3424 = 599$, a number less than 856; 4, therefore, is the correct quotient figure. The second partial dividend, moreover, is $5990 + 2$, or 5992.

8 hundreds are contained in 59 hundreds 7 times, with a considerable remainder. It is probable, therefore, that 856 will be contained in 5992 7 times. $856 \times 7 = 5992$; 7, therefore, is the correct quotient figure, and the divisor is exactly contained in the dividend.

Each partial dividend is one order lower than the preceding partial dividend; therefore, each quotient figure must be one order lower than the preceding quotient figure. The total quotient, therefore, is 7 units + 4 tens, or 47.

Ex. 2. Explain the solution of Ex. 2.

Ex. 3. Explain the solution of Ex. 3.

NOTE. The process followed in solving the exercises under Art. 27 is called SHORT DIVISION, while that employed with the exercises under Art. 29 is called LONG DIVISION. It is evident that the only difference between the processes is that in long division the intermediate results are written out, while in short division only the final result is expressed.

To Prove Multiplication and Division.

When a product and a multiplier are given how may the multiplicand be found?

When a product and a multiplicand are given how may the multiplier be found?

Give, then, two methods for proving a multiplication.

What relation has a dividend to a divisor and a quotient?

How, then, when a divisor and a quotient are given may a dividend be found?

Give, then, a method for proving a division.

What must be done if there is a remainder?

NOTE. In Ex. 44 write the part of the dividend left after division as a remainder; in Ex. 45 express it as a fraction; and in Ex. 46 extend the division to two decimal places.

Ex. 44.

Divide

1. 8463233 by 6593
2. 3002963 by 2645
3. 2906583 by 3642
4. 4296049 by 4706
5. 5400995 by 2649
6. 2460957 by 1097
7. 3279093 by 6435
8. 8324426 by 4465
9. 7554653 by 4445
10. 3556457 by 5455
11. 4637889 by 4776
12. 4678555 by 4677
13. 5789503 by 7804
14. 5900867 by 6987
15. 4790569 by 7766
16. 44690842 by 456
17. 45566825 by 635
18. 32590026 by 409
19. 20345740 by 564
20. 26908819 by 605
21. 65437613 by 255
22. 53208674 by 299
23. 37540709 by 644
24. 84904657 by 909
25. 55778688 by 756
26. 67969063 by 304
27. 64578099 by 799
28. 68869005 by 632
29. 58976708 by 895
30. 28858978 by 897

Ex. 45.

Find the ratio of

1. 1354 to 31
2. 2895 to 44
3. 5752 to 78
4. 6425 to 75
5. 3095 to 45
6. 2954 to 79
7. 2669 to 46
8. 5577 to 27
9. 2709 to 48
10. 9057 to 79
11. 9644 to 28
12. 5778 to 16
13. 4589 to 17
14. 6798 to 65
15. 2678 to 37
16. 4556 to 44
17. 3578 to 45
18. 4784 to 45
19. 3508 to 57
20. 1457 to 79
21. 2748 to 55
22. 3678 to 75
23. 3866 to 76
24. 5619 to 63
25. 5009 to 49
26. 3567 to 54
27. 6455 to 37

Ex. 46.

1. $78000476 \div 5622 = ?$
2. $46664063 \div 6170 = ?$
3. $59962047 \div 9015 = ?$
4. $95490075 \div 4670 = ?$
5. $86469827 \div 5987 = ?$
6. $90807060 \div 4059 = ?$
7. $12481632 \div 2450 = ?$
8. $69905064 \div 56908 = ?$
9. $45479977 \div 99479 = ?$
10. $57904679 \div 57943 = ?$
11. $64279969 \div 90705 = ?$
12. $62770407 \div 85739 = ?$
13. $19603048 \div 80402 = ?$
14. $36439005 \div 59797 = ?$

31. Principles of Multiplication and Division.

A man whose property is in uncertain investments, and whose relatives are scattered over different parts of the world, leaves a will which divides all his property equally among all his relatives. An estimate is made of the probable value of the property and of the probable number of relatives.

From these estimates, by what process can the probable amount to be received by each relative be obtained?

The property to be divided is what element of the operation?

The number representing the number of relatives?

The property to be received by each relative?

1. Suppose that the value of the property is found to be twice the estimated value.

How will the amount received by each relative compare with the estimated amount?

Multiplying the dividend has, then, what effect on the quotient?

2. Suppose that the value of the property is found to be only one-half the estimated value?

How will the amount received by each relative compare with the estimated amount?

Dividing the dividend has, then, what effect on the quotient?

3. Suppose that the number of relatives is found to be twice the estimated number.

How will the amount received by each relative compare with the estimated amount?

Multiplying the divisor has, then, what effect on the quotient?

4. Suppose that the number of relatives is found to be only one-half the estimated number.

How will the amount received by each relative compare with the estimated amount?

Dividing the divisor has, then, what effect on the quotient?

5. Suppose that the amount of property is found to be twice the estimated amount, and the number of relatives to be twice the estimated number.

How will the amount received by each relative compare with the estimated amount?

Multiplying both dividend and divisor has, then, what effect on the quotient?

6. Suppose that the amount of property is found to be only one-half the estimated amount, and the number of relatives to be only one-half the estimated number.

How will the amount received by each relative compare with the estimated amount?

Dividing both dividend and divisor by the same number has, then, what effect on the quotient?

The boys of a certain school determine to secure a base-ball outfit. An estimate is made of the cost of the outfit and of the probable number of boys who will contribute to its purchase.

From these estimates, by what process can the probable amount to be contributed by each boy be obtained?

The cost of the outfit is what element of the operation?

The number of boys?

The cost to each boy?

1. Explain two ways by which the cost to each boy may be twice the estimated cost.

In what two ways, then, can a quotient be multiplied?

2. Explain two ways by which the cost to each boy may be only one-half the estimated cost.

In what two ways, then, can a quotient be divided?

3. Explain how the cost to each boy may be exactly the estimated cost.

(1) Although the cost of the outfit is twice the estimated cost.

(2) Although the cost of the outfit is only one-half the estimated cost.

(3) Although the number of boys is twice the estimated number.

(4) Although the number of boys is only one-half the estimated number.

How, then, can we counteract the effect on the quotient

(1) Of multiplying the dividend by a number?

(2) Of dividing the dividend by a number?

(3) Of multiplying the divisor by a number?

(4) Of dividing the divisor by a number?

For convenience of reference, we express the preceding laws in the following

PRINCIPLES OF DIVISION.

1. Multiplying the dividend multiplies the quotient.

2. Dividing the dividend divides the quotient.

3. Multiplying the divisor divides the quotient.

4. Dividing the divisor multiplies the quotient.

5. Multiplying both dividend and divisor does not change the quotient.

6. Dividing both dividend and divisor does not change the quotient.

The preceding principles may be expressed in the following

CONDENSED PRINCIPLES.

1. A change in the dividend, by multiplication or division, produces a like change in the quotient.

2. A change in the divisor, by multiplication or division, produces an opposite change in the quotient.

3. A like change in both dividend and divisor, by multiplication or division, does not change the quotient.

A man about to visit a neighboring city decides to make extensive additions to his library. He estimates the number of books he will need to purchase and the average price of each book.

From these estimates, by what process can he determine the probable amount to be expended?

The cost of each book is what element of the operation?

The number of books to be purchased?

The total cost?

1. Suppose that the average cost of each book is twice the estimated cost.

How will the total cost compare with the estimated cost?

Suppose that the number of books that he decides to purchase is twice the estimated number.

How will the total cost compare with the estimated cost?

Multiplying either factor has, then, what effect upon the product?

2. Suppose that the cost of each book is but one-half the estimated cost.

How will the total cost compare with the estimated cost?

Suppose that the number of books he decides to purchase is but one-half the estimated number.

How will the total cost compare with the estimated cost?

Dividing either factor has, then, what effect on the product?

3. Suppose that the cost of each book is twice the estimated cost, but that he decides to purchase only one-half the estimated number; or suppose that the cost of each book is only one-half the estimated cost, but that he decides to purchase twice the estimated number. In either case, how does the total cost compare with the estimated total cost?

Multiplying one factor and dividing the other by the same number has, then, what effect upon the product?

For convenience of reference we express the preceding laws in the following

PRINCIPLES OF MULTIPLICATION.

1. Multiplying either factor multiplies the product.

2. Dividing either factor divides the product.

3. Multiplying one factor and dividing the other by the same number does not change the product.

32. To Multiply an Integer by 10, 100, etc.

In the number 675, the 5 signifies what? the 7? the 6?

Annex a 0 to 675, thus changing it to 6750.

How does the position of the 5 compare with its previous position?

How does the position of the 7 compare with its previous position?

How does the position of the 6 compare with its previous position?

How does the position of each figure compare with its previous position?

How, then, does the value of each figure compare with its previous value?

How, then, does the value of the entire number compare with its previous value?

What effect would the addition of two 0's to an integral number have

Upon the position of each figure?
Upon the value of each figure?
Upon the value of the entire number?

Explain in the same way the several effects of the addition to an integral number

Of three 0's.	Of five 0's
Of four 0's.	Of six 0's.

Give, then, a rule for multiplying an integral number

By 10.	By 1,000.	By 100,000.
By 100.	By 10,000.	By 1,000,000.

In using each of these multipliers, how does the number of 0's added to the multiplicand compare with the number of 0's in the multiplier?

Give, then, a general rule for multiplying by 10, 100, 1000, etc.

33. To Multiply a Decimal by 10, 100, etc.

In the number 6.75. the 5 represents what? the 7? the 6?

Remove the decimal one place to the right, thus changing 6.75 to 67.5.

How does the position of the 5 compare with its previous position?

How does the position of the 7 compare with its previous position?

How does the position of the 6 compare with its previous position?

How does the position of each figure compare with its previous position?

How, then, does the value of each figure compare with its previous value?

How, then, does the value of the entire decimal compare with its previous value?

What effect in a decimal would the removal of the point two places to the right have

Upon the position of each figure?
Upon the value of each figure?
Upon the value of the entire decimal?

Explain in the same way the several effects upon a decimal of removing the decimal point

Three places to the right.	Five places to the right.
Four places to the right.	Six places to the right.

Give, then, a rule for multiplying a decimal

By 10.	By 1,000.	By 100,000.
By 100.	By 10,000.	By 1,000,000.

In using each of these multipliers, how does the number of places that the decimal point is removed to the right in the multiplicand compare with the number of 0's in the multiplier?

Give, then, a general rule for multiplying a decimal by 10, 100, 1000, etc.

34. To Divide a Number by 10, 100, etc.

In the number 675, the 5 represents what? the 7? the 6?

Remove one place to the left the decimal point, which is understood at the right of the 5, thus changing 675 to 67.5.

How does the position of the 5, with reference to the decimal point, compare with its previous position?

How does the position of the 7 compare with its previous position? the position of the 6? the position of each figure?

How, then, does the value of each figure compare with its previous value?

How, then, does the value of the entire number compare with its previous value?

What effect would the removal of the decimal point two places to the left have

Upon the position of each figure?
Upon the value of each figure?
Upon the value of the entire number?

Explain in the same way the several effects from removing the decimal point

Three places to the left.	Five places to the left.
Four places to the left.	Six places to the left.

How, then, can we divide a number

By 10?	By 1,000?	By 100,000?
By 100?	By 10,000?	By 1,000,000?

In using each of these divisors, how does the number of places that the decimal point is removed to the left compare with the number of 0's in the divisor?

Give, then, a general rule for dividing by 10, 100, 1,000, etc.

		SOLUTIONS.
Ex. 1.	Multiply 5849 by 1000.	(1) 5849^{000}
Ex. 2.	Multiply 58.49 by 1000.	(2) 58.49^{0}
Ex. 3.	Divide 58.49 by 1000.	(3) $\cdot^{0}58.49$
Ex. 4.	Divide 5849 by 1000.	(4) 5·849

EXPLANATIONS.

Ex. 1. We are to multiply 5849 by 1000. To do this, we simply annex three 0's to the multiplicand, as by so doing we increase the value of each figure a thousand-fold, and so increase the value of the number a thousand-fold.

Annexing three 0's to the multiplicand, we have as our product 5849000.

Ex. 2. We are to multiply 58.49 by 1000. To do this we simply move the decimal point three places to the right, as by so doing we increase the value of each figure a thousand-fold, and so increase the value of the decimal a thousand-fold.

Removing the decimal point three places to the right, first annexing a 0 that we may be able to remove it the required number of places, we have as our product 58490.

Ex. 3. We are to divide 58.49 by 1000. To do this we simply move the decimal point three places to the left, as by so doing we decrease the value of each figure to one thousandth its original value, and so decrease the value of the decimal to one thousandth its original value.

Removing the decimal point three places to the left, first prefixing a 0, that we may be able to remove it the required number of places, we have as our quotient .05849.

Ex. 4. We are to divide 5849 by 1000. To do this we simply remove the decimal point, which is understood at the right of the 9, three places to the left, as by so doing we decrease the value of each figure to one thousandth its original value, and so decrease the value of the number to one thousandth its original value.

Removing the decimal point three places to the left, we have as our quotient 5.849.

Ex. 47.

Multiply each of the numbers in Ex. 11 by 1,000; by 10; by 100,000. Divide by the same numbers.

Form and solve exercises of your own in which different powers of 10 are multipliers or divisors. Continue this work until you can perform it without error or hesitation.

35. To Multiply and to Divide by 25, 125, etc.

We are to multiply a number by 25. Instead we multiply it by 100.

How does our multiplier compare with the correct multiplier?

How, then, will our product compare with the correct product?

How can we change our product so as to make it correct?

Give, then, a rule for multiplying a number by 25.

We are to divide a number by 25. Instead we divide it by 100.

How does our divisor compare with the correct divisor?

How, then, will our quotient compare with the correct quotient?

How can we change our quotient so as to make it correct?

Give, then, a rule for dividing a number by 25.

Explain in the same manner the effect, also how the error thus arising may be corrected, of using

As a multiplier	As a divisor
100 instead of $12\frac{1}{2}$.	100 instead of $12\frac{1}{2}$.
100 instead of $33\frac{1}{3}$.	100 instead of $33\frac{1}{3}$.
100 instead of $16\frac{2}{3}$.	100 instead of $16\frac{2}{3}$.
100 instead of 50.	100 instead of 50.
100 instead of 20.	100 instead of 20.
10 instead of $2\frac{1}{2}$.	10 instead of $2\frac{1}{2}$.
10 instead of $1\frac{1}{4}$.	10 instead of $1\frac{1}{4}$.
10 instead of $3\frac{1}{3}$.	10 instead of $3\frac{1}{3}$.
10 instead of $1\frac{2}{3}$.	10 instead of $1\frac{2}{3}$.
1000 instead of 250.	1000 instead of 250.
1000 instead of 125.	1000 instead of 125.
1000 instead of $333\frac{1}{3}$.	1000 instead of $333\frac{1}{3}$.
1000 instead of $166\frac{2}{3}$.	1000 instead of $166\frac{2}{3}$.
1000 instead of 500.	1000 instead of 500.
1000 instead of 200.	1000 instead of 200.

Give, then, a rule

For multiplying by $12\frac{1}{2}$.	For dividing by $12\frac{1}{2}$.
For multiplying by $33\frac{1}{3}$.	For dividing by $33\frac{1}{3}$.
For multiplying by $16\frac{2}{3}$.	For dividing by $16\frac{2}{3}$.
For multiplying by 50.	For dividing by 50.
For multiplying by 20.	For dividing by 20.
For multiplying by $2\frac{1}{2}$.	For dividing by $2\frac{1}{2}$.
For multiplying by $1\frac{1}{4}$.	For dividing by $1\frac{1}{4}$.
For multiplying by $3\frac{1}{3}$.	For dividing by $3\frac{1}{3}$.
For multiplying by $1\frac{2}{3}$.	For dividing by $1\frac{2}{3}$.
For multiplying by 125.	For dividing by 125.
For multiplying by 250.	For dividing by 250.

For multiplying by $333\frac{1}{3}$ For dividing by $333\frac{1}{3}$.
For multiplying by $166\frac{2}{3}$. For dividing by $166\frac{2}{3}$.
For multiplying by 500. For dividing by 500.
For multiplying by 200. For dividing by 200.

		SOLUTIONS.			
Ex. 1.	Multiply 697 by 25.	(1)	(2)	(3)	(4)
Ex. 2.	Divide 645 by $33\frac{1}{3}$.	17425	19.35	211.425	5.8416
Ex. 3.	Multiply 84.57 by $2\frac{1}{2}$.				
Ex. 4.	Divide 9.736 by $1\frac{2}{3}$.				

EXPLANATIONS.

Ex. 1. We are to multiply 697 by 25.

We first multiply by 100, by mentally annexing two o's to our multiplicand. We thus obtain as our product 69700. But in multiplying by 100 instead of by 25 we use a multiplier 4 times too large. Our product, therefore, is 4 times too large, and to obtain the correct product we must divide by 4. Dividing 69700 by 4, we have as the correct product 17425.

Ex. 2. We are to divide 645 by $33\frac{1}{3}$.

We first divide by 100, by mentally removing the decimal point, which is understood at the end of the number, two places to the left. We thus obtain as our quotient 6.45. But in dividing by 100 instead of by $33\frac{1}{3}$, we use a divisor 3 times too large. Our quotient, therefore, is only one-third large enough, and to obtain the correct quotient we must multiply by 3. Multiplying by 3, we have as the correct quotient 19.35.

Explain the solutions of Exs. 3 and 4.

NOTE 1. Ordinarily as convenient a way to multiply or divide by 50, 20, 500, or 200, is to multiply or to divide directly. In certain cases, however, to which attention will be called later, there is an advantage in following the indirect method explained in this article.

NOTE 2. When a product by 10, 100, etc., is not divisible by 3, 4, etc. it may be desired to express the indivisible part as a remainder instead of as a fractional part of the quotient. This, evidently, may be done by dividing both dividend and divisor by such a number as will make the divisor equal to the original divisor. Thus, in the second exercise the fractional part of the quotient is .35, or $35 \div 100$. $35 \div 100$ is equal to $11\frac{2}{3} \div 33\frac{1}{3}$, the original divisor. The true remainder, therefore, in Ex. 2 is $11\frac{2}{3}$.

Ex. 48.

Multiply each number in the odd exercises in Exercise 20 by 25, by $166\frac{2}{3}$, by $3\frac{1}{3}$. Divide each number in the even exercises by $33\frac{1}{3}$, by 125, by $1\frac{2}{3}$.

36. To Multiply and to Divide by a Composite Number.

We are to multiply a number by 28. Instead we multiply by 4.

How does our multiplier compare with the correct multiplier?

How, then, will our product compare with the correct product?

How can we change our product so as to make it correct?

Give, then, a special rule for multiplying by 28.

We are to divide a number by 28. Instead we divide by 4.

How does our divisor compare with the correct divisor?

How, then, will our quotient compare with the correct quotient?

How can we change our quotient so as to make it correct?

Give, then, a special rule for dividing by 28.

Explain in the same way the error produced, and how the error thus produced may be corrected, through multiplying

By 5 instead of by 25.	By 7 instead of by 56.
By 6 instead of by 42.	By 6 instead of by 54.
By 6 instead of by 48.	By 7 instead of by 63.

Explain in like manner the several effects of dividing

By 5 instead of by 25.	By 7 instead of by 56.
By 6 instead of by 42.	By 6 instead of by 54.
By 6 instead of by 48.	By 7 instead of by 63.

Give, then, a special rule for

Multiplying by 25.	Dividing by 25.
Multiplying by 42.	Dividing by 42.
Multiplying by 48.	Dividing by 48.
Multiplying by 56.	Dividing by 56.
Multiplying by any Composite Number.	Dividing by any Composite Number.

* * *

We are to divide 94367 by 35. We first divide by 5.

What is the integral part of the quotient? $18873\frac{2}{5}$

What is the fractional part? $2696\frac{7}{35}$

We divide our first quotient by 7.

What is the remainder after the fourth division?

1 is how many fifths?

5 fifths plus 2 fifths are how many fifths?

$\frac{7}{5}$, as we have previously learned, is an expression of division.

Which number in the expression is the dividend?

Which number is the divisor?

What other divisor have we?

What do we do with the two divisors?

What expression, then, represents the quotient of $\frac{7}{5} \div 7$?

Give, then, a rule for dividing a mixed number by an integer.

		(1)	(2)	(3)	(4)
Ex. 1.	Multiply 17059 by 45.	153531	$5409\frac{3}{8}$	5792	$380\frac{7}{9}$
Ex. 2.	Divide 43275 by 56.	767655	$772\frac{43}{56}$	34752	$47\frac{43}{72}$
Ex. 3.	Multiply 724 by 48.				
Ex. 4.	Divide 3427 by 72.				

EXPLANATIONS.

Ex. 1. We are to multiply 17059 by 45.

We first multiply 17059 by 9. The product is 153531. But in using 9 as a multiplier instead of 45 we use a multiplier only one-fifth large enough. Our product, therefore, is only one-fifth large enough, and to obtain the correct product we must multiply by 5. Multiplying 153531 by 5 we have as our final product 767655.

Ex. 2. We are to divide 43275 by 56.

We first divide 43275 by 8. Our quotient is $5409\frac{3}{8}$. But in using 8 as a divisor instead of 56 we use a divisor only one-seventh large enough. Our quotient, therefore, is 7 times too large, and to obtain the correct quotient we must divide by 7.

In dividing $5409\frac{3}{8}$ by 7 we have after the third division a remainder of $5\frac{3}{8}$. 1 integer equals 8 eighths, 5 integers equal 40 eighths; and 40 eighths plus 3 eighths equal 43 eighths, or $\frac{43}{8}$. $\frac{43}{8}$ is an expression of division, 43 being the dividend of the expression and 8 the divisor. Therefore, in dividing $\frac{43}{8}$ by 7 we have a dividend, 43, and two divisors, 8 and 7. 43 is divisible by neither divisor, therefore we retain our dividend unchanged and multiply together the divisors 8 and 7. The result thus obtained, which forms the fractional part of our quotient, is $\frac{43}{56}$, and our total quotient is $772\frac{43}{56}$.

Ex. 3. Explain the solution of Ex. 3.

Ex. 4. Explain the solution of Ex. 4.

NOTE. It is evident that in Exs. 2 and 4 the numerators of the fractional part of each quotient might have been written as remainders. When the remainder is expressed as a fractional part of the quotient the fraction should be reduced to its lowest terms by dividing the numerator and denominator by their common factors.

37. To Multiply when there are 0's at the Right of the Multiplier and Multiplicand.

$$\begin{array}{r} 162 \\ 108 \\ \hline 1242000 \end{array}$$

We are to multiply 2700 by 460.

Into what two convenient factors can 2700 be separated?

Into what two 460?

What two factors do we first multiply together?

What is their product?

What is the remaining factor of 2700?
What is the remaining factor of 460?
100 multiplied by 10 gives what product?
1242 multiplied by 1000 gives what product?
2700 multiplied by 460 gives, then, what product?
Give, then, a rule for multiplying when there are 0's at the right of the multiplier or multiplicand.

SOLUTIONS.

	(1)	(2)	(3)
Ex. 1. Multiply 9600 by 40.	384000	238.8	172
Ex. 2. Multiply 79.6 by 2700.		2149.20.	301
Ex. 3. Multiply 4300 by 74.			318200

EXPLANATIONS.

Ex. 1. We are to multiply 9600 by 40.

The factors of 9600 are 96 and 100, and the factors of 40, 4 and 10.

We first multiply together the factors 96 and 4. The product thus obtained is 384.

To multiply 384 by the two remaining factors we annex as many 0's as are contained in those factors. There are two 0's in 100 and one in 10. We therefore annex three 0's to 384, thus obtaining as our final product 384000.

Ex. 2. We are to multiply 79.6 by 2700.

The factors of 2700 are 27 and 100. We first multiply 79.6 by 27, thus obtaining as our product 2149.2.

To multiply 2149.2 by 100, we remove the decimal point two places to the right. We thus obtain as our final product 214920.

Ex. 3. Explain the solution of Ex. 3.

38. To Divide when there are 0's at the Right of the Divisor.

We are to divide 67865 by 3700.
Into what two convenient factors do we separate our divisor?
By which factor do we first divide?
How do we divide by 100?
What is our quotient?

```
678·65 | 37
37     | 18
---
308
296
---
 12.65
```

What is the next step of the division?
What is the integral part of our quotient?
The fractional part?
By what must we multiply 37 to change it to 3700, the original divisor?
By what, then, must we multiply 12.65 to obtain the true remainder?
What, then, is the true remainder?
Give, then, a rule for dividing when there are 0's at the right of the divisor.

SOLUTIONS.

Ex. 1. Divide 9736 by 1800.

Ex. 2. Divide 42756 by 2400.

(1)		(2)	
97·36	18	427·56	24
90	5	24	17
7.36		187	
		168	
		19.56	

EXPLANATIONS.

We are to divide 9736 by by 1800.

The factors of 1800 are 18 and 100. We first divide 9736 by 100. To do this we remove the decimal point two places to the left. The quotient thus obtained is 97.36.

We next divide 97.36 by 18. Dividing we obtain 5 as the integral part of our quotient, and 7.36.÷18 as the fractional part. To change the divisor 18 to 1800, the original divisor, we multiply it by 100; therefore to obtain the true remainder 7.36 must be multiplied by 100. Multiplying 7.36 by 100 we have as our true remainder 736.

Ex. 2. Explain the solution of Ex. 2.

Ex. 49.

Divide

1. 38065 by 270	10. 79499 by 180	19. 64237 by 320
2. 45342 by 200	11. 5753 by 60	20. 9634 by 640
3. 3455 by 240	12. 936565 by 420	21. 3456 by 2700
4. 6234 by 800	13. 89547 by 600	22. 33565 by 580
5. 52696 by 2500	14. 93937 by 970	23. 57932 by 930
6. 932460 by 600	15. 63557 by 630	24. 57047 by 370
7. 889044 by 2000	16. 4979 by 490	25. 6648 by 2400
8. 5698 by 490	17. 68049 by 7600	26. 4959 by 1400
9. 597400 by 600	18. 47906 by 400	27. 2754 by 6400

Ex. 50.

Find the product of

1. 6427000 × 2700	13. 596245600 × 5800
2. 379000 × 490	14. 659695000 × 6500
3. 499500 × 900	15. 546945 × 550
4. 656000 × 2500	16. 6294400 × 640
5. 7494000 × 9800	17. 466496 × 7200
6. 660000 × 5200	18. 65954000 × 890
7. 569000 × 7200	19. 98464 × 960
8. 6756000 × 780	20. 754757000 × 6700
9. 4559000 × 540	21. 7957849 × 9800
10. 349500 × 460	22. 645978500 × 900
11. 79697900 × 750	23. 7954869 × 940
12. 25995000 × 490	24. 4327880 × 369

39. To Multiply by 147, 125255, etc.

We are to multiply a certain number by 147.
We first multiply by 7 units.
We have left 14 tens by which to multiply.
How will the product by 14 compare with the product by 7?
How, then, after obtaining the product by 7 can we obtain the product of the same multiplicand by 14?

If the product by 7 is 28, what will be the product by 14?
If the product by 7 is 91, what will be the product by 14?
If the product by 7 is 214, what will be the product by 14?

How will the order of a product by tens compare with the order of the product of the same multiplicand by units?
Where, then, should the right-hand figure of the product by the 14 tens be placed with reference to the right-hand figure of the product by the 7 units?
Give, then, a special rule for multiplying by 147.

We are to multiply a number by 125255.
We first multiply by 5.
After obtaining the product by 5, how can we obtain the product by 25?
Where shall we place the right-hand figure of the product by the 25 tens?
How will the product by 125 compare with the product by 25?
How, then, after obtaining the product by 25, can we obtain the product by 125?

How does the order of the 125 compare with the order of the 25?
How, then, will the order of the product by 125 compare with the order of the product by 25?
Where, then, should the right-hand figure of the product by 125 be placed with reference to the product by 25?
Give, then, a special rule for multiplying by 125255.

Where should the right-hand figure of a partial product be placed with reference to the right figure of a preceding partial product

If the partial multiplier is four orders higher than the preceding partial multiplier?
If it is five orders higher?
If it is six orders higher?
If it is ten orders higher?

Give then a special rule for multiplying a number

By 426.	By 600150306.
By 642.	Ry 6251252555.
By 366.	By 2561283228.
By 217.	By 2101051555.
By 721.	By 392781.

Ex. 1. Multiply 476589 by 625125255.

Ex. 2. Multiply 476 by 568.

Ex. 3. Multiply 287 by 355.

SOLUTIONS.

(1)	(2)	(3)
2382945	3808	1435
11914725	26656	10045
59573625	270368	101885
297868125		
297927820155195		

EXPLANATIONS.

Ex. 1. We are to multiply 476589 by 625125255.

We first multiply 476589 by 5. The product thus obtained is 2382945.

25 is 5 times 5. The product of 476589 by 25 will, therefore, be 5 times its product by 5, or 5 times 2382945, or 11914725. But the 25 is one order higher than the 5. The second product, therefore, will be one order higher than the first, and its right-hand figure must be written one order at the left of the right-hand figure of the first product.

125 is 5 times 25. The product of 476589 by 125 will, therefore, be 5 times its product by 25, or 5 times 11914725, or 59573625. But the 125 is two orders higher than the 25. The third product, therefore, will be two orders higher than the second, and its right-hand figure must be written two places at the left of the right-hand figure of the second product.

625 is 5 times 125. The product of 476589 by 625 will, therefore, be 5 times its product by 125, or 5 times 59573625, or 297868125. But the 625 is three orders higher than the 125. The fourth product, therefore, will be three orders higher than the third, and its right-hand figure must be written three places to the left of the right-hand figure of the third product.

Adding the partial products we have as our total product 297927820155195.

Explain the solutions of Exs. 2 and 3.

Ex. 51.

Multiply the first and each fifth multiplicand in Ex. 34 by 147; the second and each fifth by 125255 the third and each fifth by 252 63 9; the fourth and each fifth by 56 8 72; the fifth and each fifth by 6 42 84 220.

Multiply the first and each third multiplicand in Ex. 36 by a multiplier similar to 426; the second and each third by a multiplier similar to 642; the third and each third by a multiplier similar to 42 6 48.

40. To Multiply by 99, 998, etc.

Suppose that in the first problem under multiplication we had added 35 bushels 5 instead of 4 times.

How would the result have compared with the correct result?

How could we have corrected the result without repeating the work?

We are to multiply 375 by 997. Instead we multiply by 1000.
How will our product compare with the correct product?
How can we change the product so as to make it correct?
Give, then, a special rule for multiplying by 997.

We are to multiply 927 by 45. Instead, we multiply by 50.
How will our product compare with the correct product?
How can we change the product so as to make it correct?
Give, then, a special rule for multiplying by 45.
Following the same principle, give a special rule for multiplying

By 36.	By 9.	By 98.
By 54.	By 99.	By 998.
By 89.	By 999.	By 9998.

	(1)	(2)	(3)
Ex. 1. Multiply 9376 by 98.	937600	46350	349280
Ex. 2. Multiply 927 by 45.	18752	4635	8732
Ex. 4. Multiply 8732 by 39.	918848	41715	340548

EXPLANATIONS.

Ex. 1. We are to multiply 9376 by 98.

We first multiply 9376 by 100. The product thus obtained is 937600. But in multiplying by 100 we add 9376 two too many times. Our product, therefore, is too large by twice 9376, or by 18752, and to obtain the correct product we must subtract 18752. Subtracting 18752 from 937600, we have as the correct product 918848.

Explain the solutions of Exs. 2 and 3.

Ex. 52.

Use with multipliers similar to the preceding the multiplicands in Ex. 43. Multiply the first and every fifth multiplicand by 99; the second and every fifth by 998; the third and every fifth by 29; the fourth and every fifth by 9993; the fifth and every fifth by 999975.

NOTE. Observe that in multiplying by 36, 54, etc., by this process the operations are a multiplication and a subtraction in place of two multiplications and an addition. In most exercises like those given in this note, however, the preferable method is to multiply by the factors of the multiplier. Why?

41. To Multiply by 11, 111, etc.

We are to multiply 9863 by 11.

We first multiply 9863 by the 1 unit, then by the 1 ten, and then unite the two products into the total product.

9863
9863
108493

What figures do we combine to obtain the second figure of the total product?

To obtain the third figure?

The fourth figure?

The fifth and sixth figures?

Suppose that we wish to perform the preceding multiplication without bringing down the partial products.

What would be the right-hand figure of the total product?

How should we obtain the second figure of the product?

How the third figure?

How the fourth figure?

How the fifth and sixth figures?

Give, then, a special rule for multiplying a number by 11.

Ex. 1. Multiply 2965 by 11.

Ex. 2. Multiply 2454 by 11.

Ex. 3. Multiply 8947 by 11.

SOLUTIONS.

(1)	(2)	(3)
32615	26994	98417

EXPLANATIONS.

We are to multiply 2965 by 11.

Were we to write out and combine the two partial products in the usual manner, we should as the first step in finding the total product bring down the 5 as its right-hand figure.

We should next combine the 5 and the 6, then the 6 and the 9 and the left-hand figure of the preceding sum, then the 9 and the 2 and the left-hand figure of the preceding sum, and lastly the 2 and the left-hand figure of the preceding sum.

We perform the preceding processes, but without bringing down the partial products, and obtain as our total product 32615.

Explain the solutions of Exs. 2 and 3.

Ex. 43.

Multiply each multiplicand in Ex. 34 by 11.

Develop a short method for multiplying a number by 111. In doing this picture the three products as written out in the ordinary manner, and follow the reasoning employed in discovering the short method for multiplying by 11.

Develop a similar rule for multiplying by 1111. Follow the same course of reasoning as in the preceding processes.

Table Showing Area of States and Territories with Population in 1890 and 1900.

	Pop. 1900	Pop. 1890	Area
	1		
Alaska	63,441	36,500	577,390
Texas	3,048,710	2,235,523	265,780
California	1,485,053	1,208,130	158,360
Montana	243,329	132,159	146,080
N. Mexico	195,310	153,593	122,580
Arizona	122,931	59,620	113,020
Nevada	42,335	45,761	110,700
Colorado	539,700	412,198	103,925
Wyoming	92,531	60,705	97,890
Oregon	413,536	313,767	96,030
	2		
Utah	276,749	207,905	84,970
Idaho	161,772	84,385	84,800
Minnesota	1,751,394	1,301,826	83,365
Kansas	1,470,495	1,427,096	82,080
S. Dakota	401,570	328,808	77,570
Nebraska	1,068,539	1,058,910	77,510
N. Dakota	319,146	182,719	70,795
Missouri	3,106,665	2,679,184	69,415
Washington	518,103	349,390	69,180
Georgia	2,216,331	1,837,353	59,475
	3		
Michigan	2,420,982	2,093,889	58,915
Florida	528,542	391,422	58,680
Illinois	4,821,550	3,826,351	56,650
Wisconsin	2,069,042	1,686,880	56,040
Iowa	2,231,853	1,911,896	56,025
Arkansas	1,311,564	1,128,179	53,850
Alabama	1,828,697	1,513,017	52,250
No. Carolina	1,893,810	1,617,947	52,250
New York	7,268,012	5,997,853	49,170
Indian Ter.	391,960	180,389	31,400
	4		
Louisiana	1,381,625	1,118,587	48,720
Missisippi	1,551,270	1,289,600	46,810
Pennsylvania	6,302,115	5,258,014	45,215
Virginia	1,854,184	1,655,980	42,450
Tennessee	2,020,616	1,767,518	42,050
Ohio	4,157,545	3,672,316	41,060
Kentucky	2,147,174	1,858,635	40,400
Oklahoma	398,248	61,834	39,030
Indiana	2,516,462	2,192,404	36,350
Maine	694,466	661,086	33,040

	Pop. 1900	Pop. 1890	Area
	5		
South Carolina	1,340,316	1,151,149	30,570
West Virginia . . .	958,800	762,794	24,780
Maryland	1,190,050	1,042,390	12,210
Vermont	343,641	332,422	9,565
New Hampshire . . .	411,588	376,530	9,305
Massachusetts	2,805,346	2,238,943	8,315
New Jersey	1,883,669	1,444,933	7,815
Connecticut	908,355	746,258	4,990
Delaware	184,735	168,493	2,050
Rhode Island	428,556	345,506	1,250
District of Columbia . .	278,718	230,392	70

SUGGESTIONS TO PUPILS. If you have difficulty in solving problems like the third and fifth, first perform this preliminary work.

Draw two lines representing, for example, 5 and 7 units.

What is the ratio of the shorter line to the longer? of the longer to the shorter?

Draw ten pairs of lines, each pair representing a larger number of units than the preceding pair. Write the number representing the number of units in its length at the side of each line. Name the ratio of each shorter line to the corresponding longer line; of each longer line to the corresponding shorter line.

Name the ratio of a line representing 9305 units to a line representing 84970 units; the ratio of a line representing 9305 units to a line representing 1250 units. What, then, is the ratio of the area of New Hampshire to the area of Utah? of the area of New Hampshire to the area of Rhode Island?

In all cases in which the divisor is not exactly contained in the dividend, extend the indicated division to hundredths.

By density of population is signified the number of people to the square mile. Thus, in saying that the density of population of one State is twice that of another, one means that there are twice as many people to the square mile in the first State as in the second.

The expression PER CENT signifies "by the hundred." Therefore to solve the twenty-third and similar problems, first express the ratio between the quantities compared, and, secondly, carry out to two decimal places the indicated division thus obtained.

If the exact per cent cannot be obtained, carry out the division two more decimal places, thus expressing the answer as a per cent and hundredths of a per cent.

The character % is used to represent the words per cent.

Thus 0894 may be written 8.94%.

1. Find the area of the first group of states and territories.

2. Find the area of the second group.

3. Find the ratio of the area of the first group to the area of the second group.

4. Find the area of the third group.

5. Find the ratio of the area of the second group to the area of the third group.

6. Find the area of the fourth group.

7. Find the ratio of the area of the third group to the area of the fourth group.

8. Find the area of the fifth group.

9. Find the ratio of the area of the fourth group to the area of the fifth group.

10. Find the entire area of the United States.

11. Find the population of the first group in 1900.

12. Find the population of the first group in 1890.

13. Find the per cent of increase in the first group from 1890 to 1900.

14. Find the population of the second group in 1900.

15. Find the population of the second group in 1890.

16. Find the per cent of increase in the second group from 1890 to 1900.

17. Find the population of the third group in 1900.

18. Find the population of the third group in 1890.

19. Find the per cent of increase in the third group from 1890 to 1900.

20. Find the population of the fourth group in 1900.

21. Find the population of the fourth group in 1890.

22. Find the per cent of increase in the fourth group from 1890 to 1900.

23. Find the population of the fifth group in 1900.

24. Find the population of the fifth group in 1890.

25. Find the per cent of increase in the fifth group from 1890 to 1900.

26. Find the ratio of the area of the first group to the area of the fifth group.

27. Find the ratio of the population of the first group in 1900 to that of the fifth group.

28. How many states the size of Rhode Island could be made out of Texas?

29. How many people are there to a square mile in Texas?

30. How many people are there to a square mile in Rhode Island?

31. Suppose there were as many people to the square mile in Texas as Rhode Island. How many people would there be in Texas?

32. What would be the population of Nevada if its density of population were equal to that of Rhode Island?

What was the density of population in 1900 of each of the following states and territories?

33. Of Nevada?
34. Of Arizona?
35. Of Wyoming?
36. Of Idaho?
37. Of New Mexico?
38. Of Oklahoma?
39. Of South Dakota?
40. Of Oregon?
41. Of North Dakota?
42. Of Colorado?
43. Of District of Columbia?
44. Of Massachusetts?
45. Of New Jersey?
46. Of Connecticut?
47. Of New York?
48. Of Pennsylvania?
49. Of Ohio?
50. Of Maryland?
51. Of Delaware?
52. Of Illinois?

53. The water area of Arizona is 100 square miles. What is the ratio of the water area to the total area? of the water area to the land area? of the land area to the total area?

54. The water area of Maryland is 3250 square miles. Find the ratio of the water area to the total area; of the water area to the land area; of the land area to the total area.

55. The water area of New Hampshire is 300 square miles. Find the ratio of the water area to the total area; of the water area to the land area; of the land area to the total area.

56. Arrange the states and territories according to their populations in 1900, beginning with the state with the greatest population. Find the population of the first group of ten.

57. Find the population of the second group.

58. Find the population of the third group.

59. Find the population of the fourth group.

60. Find the population of the fifth group.

61. Find the total population of the United States.

62. Find the per cent of increase in each state and territory from 1890 to 1900. Arrange the states and territories in groups of ten on this basis, beginning with the state with the greatest der cent of increase.

Summary of Definitions.

Quantity. Any thing that can be definitely measured.

Unit of Quantity. A quantity employed as the standard of measurement.

Number. The answer to the question, How many ?

Arithmetic. The science of numbers.

Notation. The process of expressing numbers in figures.

Numeration. The process of expressing numbers in words.

Addend. A given number to be combined with other given numbers.

Addition. The ordinary process of combining two or more given numbers.

Sum or **Amount**. The result of an addition.

Minuend. A given whole from which a given part called the **Subtrahend** is to be taken to find the **Remainder**.

A given larger number to be compared with a given smaller number called the **Subtrahend** to find the **Difference**.

Subtraction. The process of finding a Difference or a Remainder.

Multiplicand. A term applied to one of two or more equal addends.

Multiplier. A number showing the number of equal addends.

Multiplication. A shortened process of combining two or more equal addends.

Product. The result of a multiplication.

Factors. Numbers which multiplied together will produce a certain product.

Prime Number. A number that cannot be separated into factors.

Composite Number. A number that can be separated into factors.

Power. The product of two or more equal factors.

Square. The product of two equal factors.

Cube. The product of three equal factors.

Exponent. A figure showing to what power a number is to be raised.

Dividend. A given number to be separated into a given number of equal parts.

A given number to be measured by another given number.

Divisor. A number showing into how many equal parts the dividend is to be separated.

A given number by which another given number is to be measured.

Division. A shortened process of finding how many given equal numbers must be combined to produce another given number.

Quotient. The result of a division.

Ratio. An expression representing in a simple form the relation in magnitude of a given dividend to a given divisor.

Root. One of the equal factors of a number.

Square Root. One of the two equal factors of a number.

Cube Root. One of the three equal factors of a number.

Index. A figure written in connection with a Radical Sign to show what root of a number is to be taken.

Common Fraction. An expression of division indicated by writing the dividend above a horizontal line and the divisor below.

Denominator. A term applied to the divisor of a fraction as showing the name of the equal parts into which a unit has been divided.

Numerator. A term applied to the dividend of a fraction as showing the number of equal parts that are taken.

Terms of a Fraction. The numerator and denominator.

Review Questions.

Define a multiplicand, a multiplier, a product, multiplication; a dividend, a divisor, a quotient, division.

Define a concrete number; an abstract number. Define a ratio. What kind of a number is a multiplier? a ratio?

Define a factor; a power; a root; a square; a cube.

Name and explain the following signs:

$$\times \quad \div \quad \surd \quad : \quad ::$$

Give the value of the following expressions:

$$5\times 6 \quad 42\div 7 \quad 32:8 \quad \tfrac{42}{5} \quad 7^2 \quad 8^3 \quad 3^6 \quad \sqrt{64} \quad \sqrt[3]{125}$$

Explain the multiplication of 325 by 4. Give a rule for multiplying when the multiplier is not greater than 12.

Explain the division of 7571 by 9. Give a rule for dividing when the divisor is not greater than 12.

Explain the multiplication of 756 by 84. Give a rule for multiplying when the multiplier is greater than 12.

Explain the division of 97250 by 129. Give a rule for dividing when the divisor is greater than 12.

Give the first principle of division; the second; the third; the fourth; the fifth; the sixth. Give the first condensed principle; the second; the third.

Give the first principle of multiplication; the second; the third.

Multiply 536 by 100. Give a rule for multiplying an integer by 10, 100, etc.

Multiply 53.6 by 100. Give a rule for multiplying a decimal by 10, 100, etc.

Divide 4.375 by 100. Give a rule for dividing a decimal by 10, 100, etc.

Divide 4375 by 100. Give a rule for dividing an integer by 10, 100, etc.

Multiply 876 by 33⅓. Give a rule for multiplying by 25, 125, etc.

Divide 4576 by 33⅓. Give a rule for dividing by 25, 125, etc.

Multiply 964 by 49 without performing any addition of partial products.

Give a rule for multiplying by a composite number.

Divide 92728 by 72 without using a divisor greater than 12. Give a rule for dividing by a composite number. Explain the finding of the true remainder.

Multiply 95000 by 3500. Give a rule for multiplying when there are 0's at the right of the multiplier and multiplicand.

Divide 94376 by 8400. Give a rule for dividing when there are 0's at the end of the divisor. Explain the finding of the true remainder.

Multiply 354 by 287 by such a method that only two partial products will be produced. Give a special rule for multiplying by 147, 525, 255, etc.

Multiply 457 by 99 without multiplying directly by either of the two 9's. Give a special rule for multiplying by 19, 36, 99, 999, etc.

Multiply 2345 by 11 without bringing down either partial product. Give a special rule for multiplying by 11.

What is meant by density of population? How can the density of population of any State be found?

What character may be used in place of the words 'per cent'? To how many decimal places is this character equivalent? Write each of the following decimals as a per cent:

.75; 7.5; .7; .384; .9478; .003045.

Name and define the principal terms used in Notation and Numeration; in Addition and Subtraction; in Multiplication and Division.

Factoring.

42. The Divisibility of Numbers.

Is 2 contained without a remainder in 10 units?

One ten is how many units? Is 2, then, contained in 1 ten?

Is 2 contained in 3 tens? in 7 tens ? in 746 tens? Can you name any number of tens that 2 will not divide? Why not?

Is 2 contained in 3 tens + 6 units? why? in 36? why? in 3 tens + 7 units? why not? in 37? why not? in 7 tens + 6 units? why? in 76? why? in 7 tens + 7 units? why not? in 77? why not? in 746 tens + 8 units? why? in 7468? why? in 746 tens + 9 units? why not? in 7469? why not?

The divisibility of a number by 2 depends, then, upon what figure of the number?

What numbers, then, are divisible by 2?

Is 5 contained in 10 units? in 1 ten? in 7 tens? in 703 tens? Can you name any number of tens that 5 will not divide? why not? Is 5 contained in 7 tens + 5 units? why? in 75? why? in 7 tens + 6 units? why not? in 76? why not? in 347 tens + 0 units? why? in 3470? why? in 347 tens + 1 unit? why not? in 3471? why not?

The divisibility of a number by 5 depends, then, upon what figure of the number?

What numbers, then, are divisible by 5?

Is 4 contained in 10 units? in 100 units? in 1 hundred? in 5 hundreds? in 3086 hundreds? Can you name any number of hundreds that 4 will not divide? why not? Is 4 contained in 7 hundreds + 28? why? in 728? why? in 7 hundreds + 27? why not? in 727? why not? in 45 hundreds + 72? why? in 4572? why? in 45 hundreds + 73? why not? in 4573? why not?

The divisibility of a number by 4 depends, then, upon what figures of the number?

What numbers, then, are divisible by 4?

Is 25 contained in 10 units? in 100 units? in 1 hundred? in 3 hundreds? in 8876 hundreds? Can you name any number of hundreds that 25 will not divide? why not? Is 25 contained in 604 hundreds + 75? why? in 60475? why? in 604 hundreds + 76? why not? in 60476? why not?

The divisibility of a number by 25 depends, then, upon what figures of a number?

What numbers, then, are divisible by 25?

Is 8 contained in 10 units? in 100 units? in 1000 units? in 1 thousand? in 76 thousands? Can you name any number of thousands that 8 will not divide? why not? Is 8 contained in 73 thousands + 648? why? in 73648? why? in 73 thousands + 647? why not? in 73647? why not?

The divisibility of a number by 8 depends, then, upon what figures of the number?

What numbers, then, are divisible by 8?

Is 125 contained in 10 units? in 100 units? in 1000 units? in 1 thousand? in 532 thousands? Can you name any number of thousands that 125 will not divide? why not? Is 125 contained in 67 thousand + 250? why? in 67250? why? in 67 thousands + 260? why not? in 67260? why not?

The divisibility of a number by 125 depends, then, upon what figures of a number?

What numbers, then, are divisible by 125?

* * *

We wish to ascertain whether 4827 is divisible by 9. In doing this, we will think of 1000 as $999+1$, of 100 as $99+1$, and of 10 as $9+1$.

Following this notation, what expression will represent the 4 thousands?

What expression the 8 hundreds?

What expression the 2 tens?

What expression the entire number?

$$\begin{array}{lcl} (1) & & (2) \\ 4(999+1) & \text{or} & 4(999)+4 \\ +8(99+1) & & +8(99)+8 \\ +2(9+1) & & +2(9)+2 \\ +7 & & +7 \end{array}$$

Are 9, 99, etc., divisible by 9?

Will any number of 9's, 99's, etc., be divisible by 9?

What parts, then, of the second expression are evidently divisible by 9?

Disregarding these parts what parts remain?

How do these remaining parts compare with the figures of the number?

If, then, $4+8+2+7$, or the sum of the figures of the number, is divisible by 9, what follows as to the divisibility of the number by 9?

If $4+8+2+7$ is not divisible by 9, what follows as to the divisibility of the number by 9?

Give, then, a rule for ascertaining the divisibility of a number by 9.

Is 4827, then, divisible by 9?

By the process that was followed with 4827, determine the divisibility by 9 of each of the following numbers:

72	164	56788	973846
85	702	30474	408228

NOTE. In determining the divisibility of the figures of a number by 9, the figures 0 and 9, and figures whose sum equals 9, need not be

added. Thus, the figures in Ex. 2 may be thought of as $9+(6+3)+(5+4)$, or as 3 9's, and the figures in Ex. 1 as $(3+6)+(5+4)+3$, or as two 9's + 3.

We wish to ascertain whether 63145 is divisible by 3. In doing this we will think of 10000 as $9999+1$, of 1000 as $999+1$, of 100 as $99+1$, and of 10 as $9+1$.

Following this notation, what expression will represent the number?

Are 9, 99, 999, 9999, etc., divisible by 3?

Will any number of times 9, 99, etc., be divisible by 3?

What parts, then, of the second expression are evidently divisible by 3?

(1)	(2)
$6(9999+1)$	or $6(9999)+6$
$+3(999+1)$	$+3(999)+3$
$+(99+1)$	$+(99)+1$
$+4(9+1)$	$+4(9)+4$
$+5$	$+5$

Disregarding these parts, what parts remain?

How do these parts compare with the figures of the number?

If, then, $6+3+1+4+5$, or the sum of the figures of the number, is divisible by 3, what follows as to the divisibility of the number by 3?

If the sum of the figures of the number is not divisible by 3, what follows as to the divisibility of the number by 3?

Give, then, a rule for ascertaining the divisibility of a number by 3.

By the process that was followed with 63145, determine the divisibility by 3 of each of the following numbers:

87	346	9642	63022	584637
94	831	8057	84312	250682

* * *

If a number is divisible by 3 and by 7, what would you say as to the divisibility of the number by 7 after the factor 3 has been removed from it?

Dividing a number by 3 and dividing the quotient thus produced by 7 is equivalent to dividing the original number by what one number?

If, then, a number is divisible by each of two factors which are prime to each other, what follows as to its divisibility by the product of the two factors?

NOTE. In speaking of two numbers as prime to each other, we mean that there is no number which will divide both of them without a remainder. Thus, while 8 and 25 are composite numbers, there is no number which will exactly divide both of them, and they are prime to each other.

If a number is divisible by 4 and by 6, is the quotient obtained by dividing by 4 necessarily divisible by 6? why not?

If, then, a number is divisible by each of two factors which are not prime to each other, what would you say as to its divisibility by the product of the two factors?

We wish to ascertain whether 24541 is divisible by 11. In doing this, we will think of 10 as $11-1$, of 1000 as $1001-1$, of 100 as $99+1$, and of 10000 as $9999+1$. That is, we will think of each even power of 10 as a number one less than that power, plus 1; and of each odd power as a number one more than the power, minus 1.

Following this notation what expressions will represent the number?

Are 99, 9999, and all other numbers composed solely of an even number of 9's divisible by 11?

What parts, then, of the second expression are evidently divisible by 11?

Disregarding these parts, what parts remain?

$$\begin{array}{lll}
\text{(1)} & \text{(2)} & \\
2(9999+1) & \text{or } 2(9999) & +2 \\
+4(1001-1) & +4(1001) & -4 \\
+5(99+1) & +5(99) & +5 \\
+4(11-1) & +4(11) & -4 \\
+1 & & +1
\end{array}$$

In the expressions we use to represent the odd powers of 10, will the number of 0's be even or odd?

From a number whose first and last figures are 1's and whose intermediate figures are an even number of 0's, subtract 11. What will be the right-hand figure of the remainder?

What will the remaining figures be?

How will the number of 9's compare with the number of 0's in the minuend?

Will, then, the number of 9's be even or odd?

What follows, then, as to the divisibility of the remainder by 11?

How does the minuend compare in magnitude with the remainder?

What follows, then, as to the divisibility of the minuend by 11?

What additional parts, then, of the second expression are divisible by 11?

Disregarding these parts, what parts remain?

How do these remaining parts compare with the figures of the number?

What sign is before the last and each alternate figure?

Before the figure next to the last and each alternate figure?

How, then, shall we find the value of the remaining parts of the second expression?

If, then, the difference between the sum of the odd and the sum of the even figures of a number is divisible by 11, what follows as to the divisibility of the number by 11?

If this difference is not divisible by 11, what follows as to the divisibility of the number by 11?

Give, then, a rule for ascertaining the divisibility of a number by 11.

Is, then, 24541 divisible by 11?

Test your conclusion by performing the division.

By the process that was followed with 24541, determine the divisibility by 11 of each of the following numbers:

143	819	4939	8195	38863
467	891	6943	9305	64075

Summary of Principles.

Summarize the preceding principles by completing the following statements:

A number is divisible by 2 or by 5 if its * * * * * * * * is divisible by 2 or by 5.

A number is divisible by 4 or by 25 if the number represented by its * * * * * * * * is divisible by 4 or by 25.

A number is divisible by 8 or by 125 if the number represented by its * * * * * * * * is divisible by 8 or by 125.

A number is divisible by 3 or by 9 if * * * * * * * * is divisible by 3 or by 9.

A number is divisible by 11 if * * * * * * * * is divisible by 11.

If a number is divisible by each of two factors * * * * * * * * it is divisible by the product of the factors.

Ex. 1. Is 35463 divisible by 9?
Ex. 2. Is 93456 divisible by 9?
Ex. 3. Is 985736 divisible by 4?
Ex. 4. Is 73228 divisible by 11?
Ex. 5. Is 768345 divisible by 3?
Ex. 6. Is 975432 divisible by 8?

EXPLANATIONS.

We are to ascertain, without performing the division, whether 35463 is divisible by 9.

In doing this, we think of 10 as $9 + 1$, of 100 as $99 + 1$, of 1000 as $999 + 1$, and of 10,000 as $9999 + 1$. Following this notation, we have as an expression for the number $3(9999 + 1) + 5(999 + 1) + 4(99 + 1) + 6(9 + 1) + 3$, and as a second expression $3(9999) + 3 + 5(999) + 5 + 4(99) + 4 + 6(9) + 6 + 3$.

Any number of 9's 99's, etc., are divisible by 9, therefore those parts of the second expression containing these numbers as factors need not be regarded.

The remaining parts, 3, 5, 4, 6, and 3, are the figures of the number. $(5 + 4) + (6 + 3) + 3$, the sum of these figures, is not divisible by 9, therefore the second expression for the number is not divisible by 9, and the number itself is not divisible by 9.

Ex. 2. Explain the solution of Ex. 2.

Ex. 3. We are to ascertain whether 985736 is divisible by 4.

985736 is a certain number of hundreds plus 36.

Any number of hundreds is divisible by 4. The only question, therefore, is as to the divisibility of the 36 by 4. 36 is divisible by 4, and the number is divisible by 4.

Ex. 4. We are to ascertain whether 73228 is divisible by 11.

In doing this, we think of the even powers of 10 as 99, 9999, etc., plus 1, and of the odd powers as 11, 1001, 100001, etc., minus 1. Following this notation, we have as an expression for the number $7(9999+1)+3(1001-1)+2(99+1)+2(11-1)+8$, and as a second expression $7(9999)+7+3(1001)-3+2(99)+2+2(11)-2+8$

Any number of 99's 9999's etc., are divisible by 11, therefore those parts of the second expression containing these numbers as factors need not be regarded.

Any number consisting of an initial and a final 1 and an even number of intermediate 0's is 11 more than a number consisting of a final 0 and an even number of 9's. Such a number, therefore, is divisible by 11, and those parts of the second expression containing 11 and 1001 as factors need not be regarded.

The remaining parts, 7, 3, 2, 2, and 8, are the figures of the number. From the nature of the expression, the last figure, with each alternate preceding figure, is preceded by the plus sign, and the next to the last, with each alternate preceding figure, is preceded by the minus sign. To find the value, therefore, of these parts of the second expression, we must add the odd figures and the even figures, and find the difference between the two sums.

$7+2+8=17$. $3+2=5$. $17-5=12$. 12 is not divisible by 11. The second expression for the number, therefore, is not divisible by 11, and the number itself is not divisible by 11.

Ex. 5. Explain the solution of Ex. 5.

Ex. 6. Explain the solution of Ex. 6.

NOTE 1. Write a complete list of the numbers from 1 to 100. Cancel each number divisible by 2; by 3; by 5; by 7. What are the remaining numbers? Are any of these numbers divisible by any number? What, then, are the prime numbers between 1 and 100?

NOTE 2. To ascertain whether a number, 223 for instance, is prime or composite, test its divisibility by each of the prime numbers in order until an exact quotient is obtained or until the quotient is equal to or smaller than the divisor.

Is 223 prime or composite?

Ex. 54.

Will

3 divide 123456?	5 divide 30714?	4 divide 937236?
9 divide 62412?	11 divide 483615?	2 divide 193494?
3 divide 62513?	8 divide 935058?	6 divide 62412?
9 divide 375408?	3 divide 38065?	11 divide 492187?
6 divide 928416?	4 divide 49377?	5 divide 123345?
125 divide 69625?	125 divide 236700?	25 divide 56724?
11 divide 908215?	25 divide 967225	15 divide 273345?
8 divide 18764?	3 divide 906876?	14 divide 208614?

43. To Find the Greatest Common Divisor of Two or More Numbers.

Name three numbers that will exactly divide 24.
Name two numbers that will divide both 24 and 36.
Name the greatest number that will divide both 24 and 36.

The term **Common Divisor** is applied to a number that will exactly divide two or more given numbers, and the term **Greatest Common Divisor** to the greatest number that will exactly divide them. In the practical problems of arithmetic it is rarely, if ever, necessary to find a greatest common divisor. Several exercises are given, however, under this case, as their solution affords valuable practice in applying the principles relating to the divisibility of numbers.

* * *

$8 \times 3 = 24$

8	168	240	264
3	21	30	33
	7	10	11

We are to find the greatest common divisor of 168, 240, and 264.

By what convenient factor do we divide the given numbers?

What quotients do we obtain?

By what factor do we divide these quotients?

Will any number divide 7, 10, and 11, the second quotients?

By what factors alone, then, are 168, 240, and 264 divisible?

Dividing by 8 and by 3 is equivalent to dividing by what one number?

What, then, is the greatest common divisor of 168, 240, and 264?

Give, then, a rule for finding the greatest common divisor of two or more numbers.

Ex. 1. Find the greatest common divisor of 1260, 1836, and 2340.

SOLUTION.

$4 \times 9 = 36$

1260	1836	2340
315	459	585
35	51	65

EXPLANATION.

Ex. 1. We are to find the greatest common divisor of 1260, 1836, and 2340.

We first divide each of the three numbers by the common factor 4. The resulting quotients are 315, 459, and 585. We divide each of these quotients by the common factor 9. The resulting quotients have no common factor, therefore 4×9, or 36, is the greatest common divisior of the three numbers.

NOTE. If in doubt as to whether the given numbers have a common factor, separate one of the numbers into its prime factors, and test the divisibility of the remaining factors by each of these factors. Thus, in finding the greatest common divisor of 68, 238, and 765, we first separate 68 into the prime factors 2, 2, and 17.

Explain the remaining steps of the operation.

Ex. 55.

Find the greatest common divisor of

1.	24	36	42	54	11.	1566	3177	7830	8469
2.	85	153	187	119	12.	4939	7227	1738	6006
3.	64	48	96	136	13.	7128	2640	5232	5000
4.	128	48	90	76	14.	4175	9250	7400	9225
5.	54	152	114	972	15.	1250	3500	1750	9000
6.	144	108	153	765	16.	1130	5390	8370	3640
7.	126	161	343	933	17.	1368	2926	2660	5320
8.	80	150	340	790	18.	1764	3420	2412	7948
9.	98	147	245	672	19.	3625	4750	9250	8500
10.	162	90	216	324	20.	1150	2576	3312	4048

44. To Find the Least Common Multiple of Two or More Numbers.

Name three numbers that will exactly contain 8.
Name two numbers that will contain both 8 and 12.
Name the least number that will contain both 8 and 12.

In certain operations in fractions it becomes necessary to find a number that will contain two or more given numbers. Such a number is called a **Common Multiple**, and the least number that will contain the given numbers is called their **Least Common Multiple.** A method for finding the least common multiple is developed in the following inductive exercises.

* * *

We wish to find the least common multiple of 10 and 15. $15 \times 2 = 30$

At least how large must a number be to contain each of these numbers.

What factor is common to 10 and 15 ?

Mentally remove this factor from each number. What is the remaining factor of 10?

By what, then, must we multiply 15 to obtain a number that will contain both 15 and 10?

* * *

$168 \times 9 = 1512$

168	108
42	27
14	9

We wish to find the least common multiple of 108 and 168.

At least how large must a number be to contain each of these numbers?

As a preliminary step to ascertaining what factor is in 108 that has no corresponding factor in 168 we remove the factors that are common to both.

What factor do we first remove?

What quotients do we thus obtain?

What common factor do we remove from the quotients?

What quotients do we thus obtain?

Have these quotients any common factor?

What factor, then, is there in 108 that has no corresponding factor in 168?

By what, then, must we multiply 168 to find the least number that will contain both 108 and 168?

$168 \times 9 = ?$

What, then, is the least common multiple of 108 and 168?

Give, then, a rule for finding the least common multiple of two given numbers.

SOLUTIONS.

(1) $49 \times 5 = 245$

(2) $468 \times 7 = 3276$

468	252
117	63
13	7

(3) $340 \times 3 = 1020$

340	204
85	51
5	3

Ex. 1. Find the least common multiple of 35 and 49.

Ex. 2. Find the least common multiple of 468 and 252.

Ex. 3. Find the least common multiple of 204 and 340.

EXPLANATIONS.

Ex. 1. We are to find the least common multiple of 35 and 49.

A number to contain both 35 and 49 must be at least as large as the larger number, or as large as 49.

35 is not contained in 49. Looking for the reason for this, we observe that there is a factor 5 in 35 that has no corresponding factor in 49. To find a number, therefore, that will contain both 35 and 49 we must multiply 49 by 5. $49 \times 5 = 245$. 245, therefore, is a common multiple of 49 and 35, and as no smaller number will contain the two numbers it is their least common multiple.

Ex. 2. We are to find the least common multiple of 468 and 252.

A number to contain both 468 and 252 must be at least as large as the larger number, or as large as 468.

252 is not contained in 468. As a preliminary step in finding the reason for this we remove the factors common to the two numbers. These factors are 4 and 9.

The quotients obtained from our second division are 13 and 7. It is evident, therefore, that 252 contains a factor 7 that has no corresponding factor in 468, and that to find a common multiple of 252 and 468 we must multiply 468 by 7.

$468 \times 7 = 3276$. 3276, therefore, is a common multiple of 252 and 468, and, as no smaller number will contain them, it is their least common multiple.

Ex. 3. Explain the solution of Ex. 3.

Ex. 56.

Find the least common multiple of

1. 8 and 10	13. 8 and 12	25. 156 and 260
2. 25 and 30	14. 16 and 24	26. 324 and 180
3. 45 and 50	15. 24 and 28	27. 256 and 320
4. 4 and 6	16. 24 and 23	28. 540 and 153
5. 16 and 18	17. 12 and 15	29. 224 and 316
6. 28 and 35	18. 32 and 40	30. 816 and 456
7. 8 and 14	19. 8 and 26	31. 224 and 168
8. 12 and 18	20. 16 and 30	32. 135 and 120
9. 17 and 20	21. 22 and 80	33. 360 and 420
10. 20 and 24	22. 36 and 48	34. 960 and 588
11. 26 and 39	23. 51 and 85	35. 207 and 576
12. 24 and 42	24. 69 and 92	36. 374 and 979

* * *

$63 \times 11 \times 5 \times 3 = 10395$
945

We are to find the least common multiple of 63, 21, 33, 55, and 27.

Which of these numbers is contained in one of the remaining numbers?

How will the quotient of a number divided by 21 compare with the quotient of the same number divided by 63?

Will, then, a number that will contain 63 contain 21?

Need we, then, consider 21 in finding the least number that will contain the five numbers?

At least how large must a number be to contain the four numbers, 63, 33, 27, and 55?

Is 33 contained in 63?

Why not?

By what, then, must 63 be multiplied to obtain a number that will contain both 63 and 33?

With what expression do we compare 55?
What factor is common to 55 and to the expression 63×11?
Remove this common factor. What is the remaining factor of 55?
By what, then, must 63×11 be multiplied to obtain a number that will contain 63, 33, and 55?

With what expression do we compare 27?
What factor is common to 27 and to the expression $63 \times 11 \times 5$?
Remove this common factor. What is the remaining factor of 27?
By what, then, must $63 \times 11 \times 5$ be multiplied to obtain a number that will contain 63, 33, 55, and 27?
$63 \times 15 = ?$ $945 \times 11 = ?$ $63 \times 11 \times 5 \times 3$, then, $= ?$
What, then, is the least common multiple of the given numbers?
Give, then, a rule for finding the least common multiple of more than two numbers.

Ex. 1. Find the least common multiple of 14, 21, and 35.

Ex. 2. Find the least common multiple of 32, 72, and 100.

Ex. 3. Find the least common multiple of 100, 75, 162, and 90.

SOLUTIONS.

(1)

$= 210$
$35 \times 3 \times 2$
14 21 35

(2)

$= 7200$
$100 \times 18 \times 4$
32 72 100
18 25
8 25

(3)

$= 8100$
$162 \times 5 \times 5 \times 2$
100 75 162 90
81 45
9 5
25 54
50 81

EXPLANATIONS.

Ex. 3. We are to find the least common multiple of 14, 21, and 35.

A number to contain each of these three numbers must be as large as the largest, or as large as 35.

21 contains a factor 3 that has no corresponding factor in 35, and 14 a factor 2 that has no corresponding factor in 35. Therefore, to obtain a number that will contain 14, 21, and 35 we must multiply 35 by 3 and by 2.

$35 \times 3 \times 2 = 210$. Therefore, the least common multiple of 14, 21, and 35 is 210.

Ex. 2. We are to find the least common multiple of 32, 72, and 100.

A number to contain each of these three numbers must be at least as large as the largest, or as large as 100.

We first find the least common multiple of 72 and 100, which is 100×18.

A number to contain 32, 72, and 100 must be at least as large as 100×18. We therefore next ascertain what factor is found in 32 that has no corresponding factor in 100 or in 18. To do this, we first remove the common factors from 32 and 100. We thus find that 32 has a factor 8, that has no corresponding factor in 100. We next mentally cancel from 8 and 18 the common factor 2. We thus find that 32 contains a factor 4 that has no corresponding factor in 100 or in 18. Therefore, the least common multiple of 32, 72, and 100 is $100 \times 18 \times 4$, or 7200.

Ex. 3. We are to find the least common multiple of 100, 75, 162, and 90.

A number to contain each of these three numbers must be at least as large as the largest, or as large as 162.

We first find the least common multiple of 162 and 90, which is 162×5.

A number to contain 75, 162, and 90 must be at least as large as 162×5. We therefore next ascertain what factor is found in 75 that has no corresponding factor in 162 or in 5. To do this, we first cancel the common factors from 75 and 162. We thus find that there is a factor 25 in 75 that has no corresponding factor in 162. We next mentally cancel from 25 and 5 the common factor 5. We thus find that 75 contains a factor 5 that has no corresponding factor in 162 or in 5. Therefore, the least common multiple of 75, 162, and 90 is $162 \times 5 \times 5$.

A number to contain 100, 75, 162, and 90 must be at least as large as $162 \times 5 \times 5$. We therefore next ascertain what factor is found in 100 that has no corresponding factor in 162, 5, or 5. To do this we first cancel the common factors from 100 and 162. We thus find that 100 contains a factor, 50, that has no corresponding factor in 162. We next mentally cancel from 50 and from 5×5 the common factor 25. We thus find that 50 contains a factor 2 that has no corresponding factor in $162 \times 5 \times 5$. Therefore, the least common multiple of 100, 75, 162, and 90 is $162 \times 5 \times 5 \times 2$, or 162×50, or 8100.

NOTE. If difficulty is found in mentally cancelling, as directed in the preceding explanations, proceed as follows:

1. Write the expression for the least common multiple of the preceding numbers above and the given number below a horizontal line.

2. Cancel the common factors from the dividend and the divisor of this expression, as directed in Art. 49.

Solve in this way Ex. 2; Ex. 3.

Ex. 57.

Find the least common multiple of

1.	8	12	4	14	18	10.	3	8	18	24	16
2.	5	10	15	14	12	11.	48	84	95	108	144
3.	4	6	18	32	40	12.	50	75	60	90	100
4.	5	9	12	15	16	13.	260	238	856	70	64
5.	3	6	10	15	24	14.	45	60	75	81	23
6.	4	14	15	22	30	15.	54	44	66	90	77
7.	5	2	4	8	12	16.	128	224	316	489	369
8.	9	6	15	21	24	17.	40	99	130	144	172
9.	2	4	6	8	10	18.	27	162	252	363	121

45. To Cancel Factors Common to the Dividend and the Divisor.

The cost of 75 bushels of oats is $36. We wish to find the cost of 40 bushels.

$$\frac{36 \times \overset{2}{\cancel{50}}}{\underset{3}{\cancel{75}}} = 24$$

What expression will represent the cost of 1 bushel?

What expression, then, represents the cost of 50 bushels?

What factor is common to the dividend and the divisor of the second expression?

What operation do we perform with this factor?

What are the remaining factors of the dividend?

What is the remaining factor of the divisor?

36 ÷ 3 gives what quotient?

12 × 2 gives what product?

What, then, is the answer to the problem?

* * *

By indicating the operations in a certain problem, we obtain the accompanying expression.

$$\frac{\cancel{35} \times \cancel{3} \times \cancel{4} \times \overset{7}{\cancel{168}}}{\cancel{5} \times \underset{3}{\cancel{36}} \times \cancel{7} \times \underset{2}{\cancel{48}}} = 1\tfrac{1}{6}$$

What factor is there in the dividend that is the product of two of the factors of the divisor?

How, then, do we by a single step get rid of one factor of the dividend and of two factors of the divisor?

What are the remaining factors of the dividend?

Of the divisor?

What relation to 36 has the product of 3 and 4?

By what single step, then, can we get rid of the factors 3, 4, and 36?

What is the remaining factor of the dividend?
What are the remaining factors of the divisor?
By what large factor can we conveniently divide 48 and 168?

What is the remaining factor of the dividend?
What are the remaining factors of the divisor?
$7 \div (3 \times 2)$ gives what quotient?
What, then, is the answer to the problem?
Give, then, a rule for simplifying an expression of division when there are factors common to the dividend and the divisor.

NOTE. When a quotient figure is 1 it is sufficient to mentally record it. It is evident, however, that this factor cannot be disregarded when no other factor remains in the dividend.

SOLUTIONS.

$$(1)\ \frac{\overset{5}{\cancel{75}} \times \cancel{18} \times \cancel{2}}{\cancel{5} \times \cancel{3} \times \cancel{36}} = 5 \qquad (2)\ \frac{\cancel{9} \times \cancel{25} \times \cancel{28} \times \overset{2}{\cancel{320}}}{\cancel{7} \times \cancel{18} \times \cancel{4} \times \cancel{160} \times \cancel{25}} = 1$$

EXPLANATIONS.

Ex. 1. We are to simplify the expression $75 \times 18 \times 2) \div (5 \times 3 \times 36)$.

We observe that the product of the last two factors of the dividend is equal to the last factor of the divisor. We therefore divide the dividend of the expression by 18×2, and the divisor by 36, the equivalent of 18×2.

We also observe that the product of the first two factors of the divisor is contained in the first factor of the dividend. We therefore divide the divisor by 5×3, and the dividend by 15, the equivalent of 5×3.

15 is contained in 75 5 times. The remaining factor of the dividend, therefore, and the value of the expression, is 5.

Ex. 2. We are to simplify the expression $(9 \times 25 \times 28 \times 320) \div (7 \times 18 \times 4 \times 160 \times 25)$.

We observe that the product of 7 and 4, two of the factors of the divisor, is equal to 28, one of the factors of the dividend. We therefore divide the divisor of the expression by 7×4, and the dividend by 28, the equivalent of 7×4.

We observe that a factor, 25, is found in the dividend and in the divisor. We therefore divide both dividend and divisor by 25.

We observe that 160, a factor of the divisor, is contained in **320**,

a factor of the dividend. We therefore divide both dividend and divisor by 160.

We observe that the product of 2 and 9, the remaining factors of the dividend, is equal to 18, the remaining factor of the divisor. We therefore divide the divisor by 18, and the dividend by 9×2, the equivalent of 18.

All the remaining factors of both dividend and divisor are 1's. The expression therefore equals $1\div1$, or 1.

Ex. 58.

Find the value of the following expressions:

1.	$(32\times75)\div(64\times50)$	10.	$(40\times25)\div(8\times15\times5)$
2.	$(48\times75)\div(60\times36)$	11.	$(120\times88)\div(22\times5\times6)$
3.	$(81\times45)\div(54\times63)$	12.	$(250\times84)\div(25\times5\times12)$
4.	$(66\times36)\div(54\times26)$	13.	$(124\times75)\div(62\times5\times15)$
5.	$(35\times48)\div(42\times64)$	14.	$(88\times40)\div(11\times15\times8)$
6.	$(64\times28)\div(42\times30)$	15.	$(65\times24)\div(26\times2\times12)$
7.	$(80\times54)\div(75\times36)$	16.	$(81\times35)\div(3\times28\times9)$
8.	$(33\times18)\div(24\times25)$	17.	$(72\times95)\div(6\times19\times3)$
9.	$(70\times32)\div(48\times14)$	18.	$(84\times50)\div(75\times7\times3)$

19. $(24\times45\times84\times57)\div(3\times5\times8\times9\times12\times38)$

20. $(75\times77\times168\times54\times64)\div(11\times84\times50\times7\times27)$

21. $(720\times1100\times585\times224)\div(6\times1000\,3\times22\times4\times81)$

Ex. 59.

1. Find the prime factors of 1728; of 936; of 385; of 1683; of 756; of 2925.

(First separate each number into factors of convenient size, like 8, 9, 12, etc.; then mentally separate these factors into their prime factors.)

2. Is 2882880 divisible by 9? by 3? by 5? by 2? by 4? by 8? by 25? by 11?

3. Is 498762 divisible by 8? by 4? by 2? by 5? by 3? by 9? by 11?

4. Find the greatest common divisor of 96, 153, 432, and 738; of 18200, 7436, and 7488.

5. Find the least common multiple of 126, 855, 1089, 738, and 14157.

6. Find the least common multiple of the numbers from 1 to 10; from 11 to 16; from 16 to 22; from 26 to 31; from 32 to 36; from 37 to 42; from 43 to 48; from 49 to 54; from 55 to 60.

Common Fractions.

As explained in Art. 23, Note 13, an expression of division in which the dividend is written above the divisor is called a **Fraction.** It is evident that an application of the principles of division will enable one to multiply or to divide a fraction or to change its form without affecting its value.

When the dividend and the divisor of a fraction have no common factor, the fraction is said to be in its **Lowest Terms.**

46. To Reduce a Fraction to its Lowest Terms.

There are 128 cubic feet in a cord.
What part of a cord, then, is one cubic foot?
What part of a cord are

2 cubic feet?	27 cubic feet?	84 cubic feet?
9 cubic feet?	64 cubic feet?	116 cubic feet?

A wood-chopper cuts 84 cubic feet of wood on Monday and 116 cubic feet on Tuesday.
What part of a cord does he cut on Monday?
What part of a cord on Tuesday?

We wish to reduce the expressions $\frac{84}{128}$ and $\frac{116}{128}$ to the simplest equivalent expressions.
Give the sixth principle of division.
Explain the reduction of the first expression.
Of the second expression.
Give a rule for reducing a fraction to its lowest terms.

$$\frac{84}{128} = \frac{21}{32}$$
$$\frac{116}{128} = \frac{29}{32}$$

SOLUTIONS.

Ex. 1. Reduce $\frac{144}{256}$ to its lowest terms. $\frac{144}{256} = \frac{18}{32} = \frac{9}{16}$

Ex, 2. Reduce $\frac{720}{960}$ to its lowest terms. $\frac{720}{960} = \frac{72}{96} = \frac{3}{4}$

EXPLANATIONS.

Ex. 1. We are to reduce $\frac{144}{256}$ to its lowest terms.

Dividing both dividend and divisor by the same number does not change the value of an expression of division. We therefore divide both terms of the fraction $\frac{144}{256}$ by 8, and both terms of $\frac{18}{32}$, the fraction thus obtained, by 2. We thus obtain as the simplest expression for the given fraction $\frac{9}{16}$.

Ex. 2. Explain the solution of Ex. 2.

Ex. 60.

Reduce to their lowest terms the following fractions-

1. $\frac{18}{24}$	6. $\frac{85}{112}$	11. $\frac{125}{250}$	16. $\frac{775}{850}$
2. $\frac{35}{42}$	7. $\frac{33}{77}$	12. $\frac{548}{664}$	17. $\frac{316}{424}$
3. $\frac{27}{63}$	8. $\frac{56}{84}$	13. $\frac{158}{632}$	18. $\frac{925}{775}$
4. $\frac{40}{75}$	9. $\frac{75}{90}$	14. $\frac{715}{945}$	19. $\frac{296}{720}$
5. $\frac{64}{96}$	10. $\frac{63}{81}$	15. $\frac{432}{882}$	20. $\frac{498}{542}$

47. To Add or to Subtract Fractions having a Common Denominator.

Suppose that the chopper had cut one cord on Monday, and two cords on Tuesday.

How would we find the amount cut by him on both days?

What is the common unit to be added?

If instead, as previously stated, he cuts $\frac{21}{32}$ of a cord on Monday, and $\frac{29}{32}$ of a cord on Tuesday, what is the common unit to be added?

How many of these units does he cut

On Monday?

On Tuesday?

On both days?

Write, then, an expression indicating the amount he cuts on both days.

Give a rule for adding fractions having a common denominator.

Write an expression indicating the difference between the amount he cuts on Monday and the amount he cuts on Tuesday.

Give, then, a rule for subtracting fractions having a common denominator.

	SOLUTIONS.	
	(1)	(2)
Ex. 1. Find the sum of $\frac{7}{18}$, $\frac{5}{18}$, and $\frac{9}{18}$.	$1\frac{1}{6}$	$\frac{1}{3}$
Ex. 2. Subtract $\frac{18}{33}$ from $\frac{29}{33}$.		

EXPLANATIONS.

Ex. 1. We are to add $\frac{7}{18}$, $\frac{5}{18}$, and $\frac{9}{18}$.

These numbers have a common unit, $\frac{1}{18}$. Therefore, we proceed with our addition as though we were adding tens or units; thus, 7, 12, 21 eighteenths. Our answer, therefore, is $\frac{21}{18}$, or $1\frac{3}{18}$, or $1\frac{1}{6}$.

Ex. 2. Explain the solution of Ex. 2.

NOTE. A fraction in which the indicated operation can be performed is called an **Improper Fraction.** An improper fraction evidently can be changed to a mixed number by performing the indicated operation.

Ex. 61.

1. $\frac{3}{7}+\frac{2}{7}=?$	6. $\frac{7}{15}+\frac{4}{15}+\frac{11}{15}=?$	11. $\frac{47}{96}-\frac{28}{96}=?$
2. $\frac{7}{9}-\frac{2}{9}=?$	7. $\frac{14}{25}+\frac{7}{25}+\frac{8}{25}=?$	12. $\frac{35}{72}-\frac{29}{32}=?$
3. $\frac{32}{36}+\frac{18}{36}=?$	8. $\frac{25}{70}+\frac{19}{70}+\frac{43}{70}=?$	13. $\frac{60}{90}-\frac{25}{90}=?$
4. $\frac{19}{45}+\frac{24}{45}=?$	9. $\frac{134}{256}+\frac{138}{256}+\frac{48}{256}=?$	14. $\frac{53}{80}-\frac{29}{80}=?$
5. $\frac{35}{70}+\frac{14}{70}=?$	10. $\frac{720}{936}+\frac{450}{936}+\frac{218}{936}=?$	15. $\frac{46}{56}-\frac{28}{56}=?$

48. To Add or to Subtract Fractions not having a Common Denominator.

Suppose that the chopper had cut two cords on Monday and 256 cubic feet on Tuesday.

Could we have directly united these two amounts into one equivalent amount?

Why not?

How can we change the form of one of these amounts so as to render it capable of being united with the other?

Suppose that he had cut $\frac{21}{32}$ of a cord on Monday, and $\frac{59}{64}$ of a cord on Tuesday.

Could we have directly united these two amounts into one equivalent amount?

Why not?

What, then, must we do with fractions that have not a common denominator to render them capable of addition?

Give Prin. 5 of Division.

By what must both terms of the fraction $\frac{21}{32}$ be multiplied to change it to an equivalent fraction with the divisor 64?

$\frac{21}{32}$, then, equals how many sixty-fourths?

Give a rule for changing a fraction to an equivalent fraction with higher terms.

Suppose that the second fraction had been $\frac{59}{80}$ instead of $\frac{59}{64}$. Could we have conveniently changed $\frac{21}{32}$ to an equivalent fraction with the divisor 80?

What is the least common multiple of 32 and 80?

$\frac{21}{32}$ and $\frac{59}{80}$ can, then, be changed to equivalent fractions having what common denominator?

$\frac{21}{32}$ equals how many one hundred sixtieths?

$\frac{59}{80}$ equals how many one hundred sixtieths?

Give a rule for reducing fractions to equivalent fractions having a common denominator.

$\frac{105}{160}+\frac{118}{160}=?$ $\frac{21}{80}+\frac{59}{80}$, then, $=?$

Give a rule for adding fractions not having a common denominator.

SOLUTIONS.

Ex. 1. Add $\frac{7}{15}$, $\frac{8}{20}$, and $\frac{9}{25}$. $\frac{140}{300}+\frac{120}{300}+\frac{108}{300}=\frac{368}{300}=1\frac{17}{75}$

Ex. 2. Subtract $\frac{7}{18}$ from $\frac{23}{24}$. $\frac{69}{72}-\frac{28}{72}=\frac{41}{72}$

EXPLANATIONS.

Ex. 1. We are to add $\frac{7}{15}$, $\frac{8}{20}$, and $\frac{9}{25}$.

The first step is to change these fractions to equivalent fractions having a common denominator.

The least common multiple of the denominators 15, 20, and 25 is 300. To change the denominator 15 to 300 we must multiply it by 20. But if we multiply the divisor 15 by 20 we must also multiply the dividend 7 by 20. The product of 7 by 20 is 140; therefore, $\frac{7}{15}$ equals $\frac{140}{300}$.

Proceeding in the same way with the remaining fractions, we find that $\frac{8}{20}$ equals $\frac{120}{300}$, and that $\frac{9}{25}$ equals $\frac{108}{300}$. Adding $\frac{140}{300}$, $\frac{120}{300}$, and $\frac{108}{300}$, we find that the sum is $\frac{368}{300}$, or $1\frac{17}{75}$. Therefore, $\frac{7}{15}+\frac{8}{20}+\frac{9}{25}=1\frac{17}{75}$.

Ex. 2. Explain the solution of Ex. 2.

Ex. 62.

Read carefully the Notes at the end of this Article, and then find the value of the following expressions:

1. $\frac{3}{8}+\frac{1}{8}$	11. $\frac{5}{8}-\frac{3}{8}$	21. $\frac{23}{25}+\frac{28}{30}+\frac{31}{35}$
2. $\frac{2}{3}+\frac{3}{4}$	12. $\frac{3}{4}-\frac{2}{3}$	22. $\frac{9}{14}+\frac{7}{18}+\frac{19}{20}$
3. $\frac{1}{2}+\frac{2}{3}$	13. $\frac{2}{3}-\frac{1}{4}$	23. $\frac{23}{27}+\frac{47}{63}+\frac{58}{84}$
4. $\frac{1}{3}+\frac{3}{4}$	14. $\frac{5}{8}-\frac{1}{2}$	24. $\frac{47}{54}+\frac{35}{64}+\frac{65}{62}$
5. $\frac{3}{5}+\frac{4}{7}$	15. $\frac{5}{9}-\frac{3}{7}$	25. $\frac{43}{45}+\frac{51}{60}+\frac{47}{75}$
6. $\frac{5}{9}+\frac{3}{7}$	16. $\frac{6}{7}-\frac{3}{5}$	26. $\frac{42}{55}+\frac{62}{77}+\frac{45}{88}$
7. $\frac{7}{8}+\frac{5}{6}$	17. $\frac{37}{45}-\frac{29}{40}$	27. $\frac{54}{83}+\frac{67}{84}-\frac{85}{96}$
8. $\frac{5}{6}+\frac{4}{9}$	18. $\frac{43}{49}-\frac{37}{63}$	28. $\frac{39}{48}-\frac{23}{88}-\frac{13}{80}$
9. $\frac{5}{8}+\frac{5}{9}$	29. $\frac{35}{56}-\frac{33}{72}$	29. $\frac{93}{99}-\frac{25}{77}-\frac{23}{48}$
10. $\frac{5}{6}+\frac{3}{4}$	20. $\frac{24}{38}-\frac{45}{79}$	30. $\frac{7}{13}+\frac{20}{21}-\frac{21}{23}$

NOTE 1. A convenient arrangement of the results in an exercise in addition is shown in the accompanying solution.

$$\overset{288}{96}\times 3\times 7$$

$$\overset{198}{\frac{33}{42}}+\overset{156}{\frac{39}{72}}+\overset{249}{\frac{83}{96}}=\frac{\begin{matrix}4032\\4419\end{matrix}}{\begin{matrix}1743\\1092\\1584\end{matrix}\over 2016}=2\tfrac{387}{2016}.$$

NOTE 2. Observe that the following are the principal steps of the solution.

We find the least common multiple of the denominators to be $96\times 3\times 7$, or 288×7, or 2016.

To find by what 42 must be multiplied to produce the least common multiple, we imagine 42 to be written under the expression 96

$\times 3 \times 7$. We next mentally cancel 3×7 from the dividend and 21 from the divisor of this new expression. We next divide 96, the remaining factor of the dividend, by 2, the remaining factor of the divisor. The quotient thus obtained is 48.

We next multiply 33 by 6 and by 8, the factors of 48. The product thus obtained is 1584. Our first numerator, therefore, is 1584.

Explain the finding of the second numerator; of the third numerator. Explain the remaining steps of the solution.

49. To Add or to Subtract Mixed Numbers.

A decimal, in the sense that it represents one or more equal parts of a unit, is evidently a fraction. In distinction, fractions that are expressed by writing the dividend above and the divisor below a horizontal line are called **Common Fractions.**

The term **Mixed Number** is used in the same sense with common fractions as with decimals.

We wish to add the mixed numbers $2\frac{1}{2}$, $4\frac{2}{3}$, and $5\frac{1}{4}$.
$\frac{1}{2} + \frac{2}{3} + \frac{1}{4} = ?$
$2 + 4 + 5 = ?$
$11 + 1\frac{5}{12} = ?$
$2\frac{1}{2} + 4\frac{2}{3} + 5\frac{1}{4}$, then, $= ?$
Give, then a rule for adding mixed numbers.

We wish to subtract $2\frac{3}{4}$ from $5\frac{2}{3}$.
Change $\frac{3}{4}$ and $\frac{2}{3}$ to equivalent fractions having a common denominator. What are these fractions?
Can we subtract $\frac{9}{12}$ from $\frac{8}{12}$?

Suppose that we add $\frac{12}{12}$ to the minuend. How many integers must we add to the subtrahend to counterbalance this addition?
$\frac{8}{12} + \frac{12}{12} = ?$
$\frac{20}{12} - \frac{9}{12} = ?$
$2 + 1 = ?$
$5 - 3 = ?$
$5^2 - 2\frac{3}{4}$, then, $= ?$
Give, then, a rule for subtracting mixed numbers.

Ex. 1. Add $1\frac{3}{4}$, $2\frac{2}{3}$, and $4\frac{7}{10}$.

Ex. 2. Subtract $9\frac{9}{14}$ from $15\frac{4}{21}$.

Ex. 3. Add $25\frac{13}{40}$ and $32\frac{29}{30}$.

Ex. 4. Subtract $25\frac{9}{12}$ from $34\frac{13}{16}$.

SOLUTIONS.

$7 + \frac{45}{60} + \frac{40}{60} + \frac{42}{60} = 7 + \frac{127}{60} = 9\frac{7}{60}$

$15\frac{8}{42} - 9\frac{27}{42} = 5\frac{23}{42}$

$57 + \frac{65}{200} + \frac{116}{200} = 57\frac{181}{200}$

$34\frac{39}{48} - 25\frac{36}{48} = 9\frac{3}{48} = 9\frac{1}{16}$

Ex. 1. We are to add $1\frac{3}{4}$, $2\frac{2}{3}$, and $4\frac{7}{10}$.

$\frac{3}{4}+\frac{2}{3}+\frac{7}{10}=\frac{45}{60}+\frac{40}{60}+\frac{42}{60}$, or $\frac{127}{60}$, or $2\frac{7}{60}$. $2+1+2+4=9$. Therefore, $1\frac{3}{4}+2\frac{2}{3}+4\frac{7}{10}=9\frac{7}{60}$

Ex. 2. We are to subtract $9\frac{9}{14}$ from $15\frac{4}{21}$

$\frac{4}{21}=\frac{8}{42}$, and $\frac{9}{14}=\frac{27}{42}$. We cannot subtract $\frac{27}{42}$ from $\frac{8}{42}$, We there- add $\frac{42}{42}$ to the minuend, which addition we counterbalance by adding one integer to the subtrahend.

$\frac{50}{42}-\frac{27}{42}=\frac{23}{42}$, and $15-10=5$. Therefore, $15\frac{4}{21}-9\frac{9}{14}=5\frac{23}{42}$.

Explain the solutions of Exs. 3 and 4.

NOTE 1. The easier way to subtract the fractional part in Ex. 2 might be to subtract $\frac{27}{42}$ from $\frac{42}{42}$, and then add $\frac{8}{42}$ to the remainder. Why?

NOTE 2. For the process of reducing a mixed number to an improper fraction see Art. 36.

NOTE. 3. The accompanying solution shows a convenient form for arranging the results in addition and in subtraction of mixed numbers.

Explain the first solution: the second.

$$\begin{array}{rr} & 240 \\ 2 & \underline{319} \\ 17\frac{3}{4} & 90 \\ 19\frac{5}{6} & 100 \\ 54\frac{3}{8} & 45 \\ \underline{49\frac{7}{10}} & 84 \\ 141\frac{79}{120} & \end{array} \qquad \begin{array}{rr} 25\frac{5}{12} & 20 \\ \underline{13\frac{7}{16}} & 21 \\ 11\frac{47}{48} & \end{array}$$

Ex. 63.

Add	Subtract	
1. $3\frac{3}{4}$, $5\frac{1}{2}$, $2\frac{3}{8}$	11. $3\frac{1}{4}$ from $4\frac{1}{2}$	21. $3\frac{1}{4}-5\frac{2}{3}+4\frac{2}{3}=?$
2. $7\frac{1}{4}$, $8\frac{1}{3}$, $9\frac{1}{2}$	12. $5\frac{2}{3}$ from $9\frac{1}{4}$	22. $7\frac{1}{2}-4\frac{1}{4}+5\frac{7}{8}=?$
3. $5\frac{3}{8}$, $9\frac{3}{4}$, $8\frac{2}{3}$	13. $7\frac{3}{8}$ from $8\frac{1}{4}$	23. $8\frac{2}{3}+7\frac{3}{4}-5\frac{1}{2}=?$
4. $2\frac{1}{2}$, $3\frac{1}{4}$, $4\frac{3}{4}$	14. $5\frac{3}{4}$ from $7\frac{1}{2}$	24. $9\frac{3}{4}+4\frac{7}{8}-3\frac{1}{8}=?$
5. $5\frac{3}{4}$, $7\frac{1}{2}$, $9\frac{1}{3}$	15. $7\frac{7}{8}$ from $8\frac{3}{4}$	25. $8\frac{1}{4}-7\frac{1}{3}+3\frac{3}{4}=?$
6. $8\frac{1}{2}$, $5\frac{3}{8}$, $9\frac{3}{4}$	16. $5\frac{7}{8}$ from $6\frac{1}{4}$	26. $8\frac{1}{2}=1\frac{1}{8}-5\frac{1}{3}=?$
7. $3\frac{3}{8}$, $6\frac{5}{8}$, $8\frac{3}{4}$	17. $6\frac{1}{2}$ from $9\frac{5}{8}$	27. $6\frac{3}{4}+2\frac{1}{4}-7\frac{7}{8}=?$
8. $4\frac{5}{8}$, $5\frac{3}{8}$, $8\frac{1}{2}$	18. $5\frac{3}{4}$ from $9\frac{1}{8}$	28. $8\frac{1}{8}+6\frac{1}{4}+8\frac{1}{2}=?$
9. $6\frac{1}{2}$, $7\frac{1}{4}$, $8\frac{1}{8}$	19. $4\frac{7}{8}$ from $8\frac{3}{4}$	29. $7\frac{1}{4}+3\frac{3}{8}-1\frac{7}{8}=?$
10. $3\frac{7}{8}$, $7\frac{3}{4}$, $4\frac{1}{2}$	20. $4\frac{3}{8}$ from $7\frac{3}{4}$	30. $6\frac{1}{4}+7\frac{1}{2}-6\frac{7}{8}=?$

Ex. 64.

Add	Subtract
1. $25\frac{13}{25}$, $14\frac{37}{50}$, $34\frac{72}{75}$, $49\frac{83}{100}$	7. $40\frac{50}{60}$ from $51\frac{62}{75}$
2. $54\frac{29}{32}$, $85\frac{37}{40}$, $72\frac{43}{56}$, $36\frac{59}{42}$	8. $74\frac{39}{64}$ from $88\frac{43}{56}$
3. $65\frac{27}{39}$, $43\frac{37}{48}$, $72\frac{49}{60}$, $31\frac{23}{156}$	9. $48\frac{64}{56}$ from $60\frac{44}{63}$
4. $77\frac{45}{50}$, $81\frac{56}{80}$, $99\frac{47}{70}$, $90\frac{19}{120}$	10. $49\frac{34}{72}$ from $50\frac{81}{84}$
5. $36\frac{26}{35}$, $54\frac{25}{42}$, $28\frac{43}{49}$, $24\frac{19}{75}$	11. $62\frac{56}{63}$ from $89\frac{49}{96}$
6. $49\frac{28}{77}$. $72\frac{34}{33}$, $81\frac{35}{55}$, $43\frac{97}{105}$	12. $29\frac{65}{81}$ from $45\frac{79}{96}$

50. To Multiply a Fraction by an Integer.

We are to multiply $\frac{7}{9}$ by 3.
In the expression $\frac{7}{9}$, which number is the dividend?
Which number is the divisor?

In what two ways can the value of an expression of division be multiplied?
What is the result by the first method?
By the second method?

Suppose the multiplicand to have been $\frac{7}{8}$ instead of $\frac{7}{9}$.
Could the shorter method of multiplication have been employed?
Give, then, a general rule for multiplying a fraction by an integer.

Ex. 1. Multiply $\frac{14}{15}$ by 5.
Ex. 2. Multiply $\frac{14}{17}$ by 5.
Ex. 3. Multiply $\frac{17}{20}$ by 15.
Ex. 4. Multiply $\frac{120}{256}$ by 224.

SOLUTIONS.

(1) $4\frac{2}{3}$

(2) $\frac{70}{17} = 4\frac{2}{17}$

(3) $\frac{51}{4} = 12\frac{3}{4}$

(4)

$$\frac{\overset{15}{\cancel{120}} \times \overset{7}{\cancel{224}}}{\underset{\underset{1}{\cancel{32}}}{\cancel{256}}} = 105$$

EXPLANATIONS.

Ex. 1. We are to multiply $\frac{14}{15}$ by 5.

Dividing a divisor multiplies the value of an expression of division. Therefore, to multiply $\frac{14}{15}$ by 5 we simply divide the divisor 15 by 5. The result thus obtained is $\frac{14}{3}$, or $4\frac{2}{3}$.

Ex. 2. We are to multiply $\frac{14}{17}$ by 5.

Multiplying a dividend multiplies the value of an expression of division. Therefore, to multiply $\frac{14}{17}$ by 5 we simply multiply the dividend 14 by 5. The result thus obtained is $\frac{70}{17}$, or $4\frac{2}{17}$.

Ex. 3. We are to multiply $\frac{17}{20}$ by 15.

Multiplying a dividend multiplies the value of an expression of division. Therefore, we may multiply $\frac{17}{20}$ by 15 by multiplying the dividend by 15.

20 and 15 contain the common factor 5. Therefore, instead of directly multiplying 17 by 15, we think of the multiplication as indicated, and mentally divide 20, the divisor, and 15, a factor of the dividend, by the common factor 5. As the final step of the operation, we must multiply together 17 and 3, the remaining factors of the dividend, and divide their product, 51, by 4, the remaining factor of the divisor, The result thus obtained is $12\frac{3}{4}$.

Ex. 4. We are to multiply $\frac{120}{256}$ by 224.

Multiplying a dividend multiplies the value of an expression of

division. Therefore we may multiply $\frac{120}{256}$ by 224 by multiplying 120 by 224. We indicate this multiplication instead of directly performing it, as there are factors common to 256, the divisor, and 120 and 224, the factors of the dividend. Cancelling these common factors and performing the indicated operation we have as the answer to the exercise 105.

NOTE. As a rule, reduce the given fraction to its lowest terms before performing the multiplication.

Ex. 65.

Find the product of

1. 8 and $\frac{3}{4}$.
2. 9 and $\frac{2}{3}$.
3. 7 and $\frac{5}{8}$.
4. 5 and $\frac{2}{3}$.
5. 7 and $\frac{5}{8}$.
6. 7 and $\frac{2}{3}$.
7. 8 and $\frac{3}{4}$.
8. 6 and $\frac{7}{8}$.
9. 3 and $\frac{5}{6}$.
10. 6 and $\frac{8}{9}$.
11. 6 and $\frac{4}{7}$.
12. 4 and $\frac{7}{8}$.
13. 24 and $\frac{7}{18}$.
14. 96 and $\frac{59}{36}$.
15. 36 and $\frac{5}{12}$.
16. 42 and $\frac{7}{18}$.
17. 84 and $\frac{5}{12}$.
18. 84 and $\frac{5}{11}$.
19. 24 and $\frac{3}{16}$.
20. 72 and $\frac{39}{48}$.
21. 17 and $\frac{25}{34}$.
22. 25 and $\frac{71}{80}$.
23. 45 and $\frac{11}{15}$.
24. 18 and $\frac{14}{25}$.
25. 350 and $\frac{27}{325}$.
26. 225 and $\frac{450}{500}$.
27. 590 and $\frac{75}{125}$.
28. 456 and $\frac{332}{576}$.
29. 163 and $\frac{447}{450}$.
30. 148 and $\frac{232}{296}$.
31. 691 and $\frac{649}{701}$.
32. 244 and $\frac{1}{488}$.
33. 620 and $\frac{2}{355}$.
34. 625 and $\frac{43}{105}$.
35. 421 and $\frac{5}{337}$.
36. 469 and $\frac{421}{870}$.

51. To Multiply an Integer by a Fraction.

We are to multiply 7 by $\frac{5}{6}$.
We first multiply 7 by 5.
How does our multiplier compare with the correct multiplier?
How, then, does our product compare with the correct product?
How can we change the product so as to make it correct?
Give, then, a rule for multiplying an integer by a fraction.

Applying the preceding principles, complete the following rules:

To multiply by $\frac{5}{6}$ * * * by * * and * * * by * * .
To multiply by $\frac{11}{13}$ * * * by * * and * * * by * * .
To multiply by $\frac{23}{29}$ * * * by * * and * * * by * * .
To multiply by any fraction, * * * by * * * and * * * by * * * * .

We are to multiply 12 by $\frac{5}{6}$.
How would we multiply 12 by $\frac{1}{6}$?
$\frac{5}{6}$ is how many times $\frac{1}{6}$?
How, then, may we multiply 12 by $\frac{5}{6}$?
Give, then, a rule for multiplying an integer by a fraction when the divisor of the fraction is contained in the integer.

Assuming that the multiplicand is divisible by the denominator of the multiplier, complete the following rules:

To multiply by $\frac{5}{6}$, * * * by * * and * * * by * * .
To multiply by $\frac{11}{15}$, * * * by * * and * * * by * * .
To multiply by $\frac{23}{29}$, * * * by * * and * * * by * * .
To multiply by any fraction, * * * by * * * and * * * by * * * .

Ex. 1. Multiply 8 by $\frac{9}{13}$.
Ex. 2. Multiply 12 by $\frac{3}{4}$.
Ex. 3. Multiply 10 by $\frac{7}{15}$.
Ex. 4. Multiply 144 by $\frac{49}{120}$.

SOLUTIONS.

(1)	(2)	(3)	(4)
$\frac{72}{13}=5\frac{7}{13}$	9	$4\frac{2}{3}$	$\frac{294}{5}=58\frac{4}{5}$

EXPLANATIONS.

Ex. 1. We are to multiply 8 by $\frac{9}{13}$.

We first multiply 8 by 9. The product is 72. But in multiplying by 9 integers instead of 9 thirteenths, we use a multiplier 13 times too large. Our product, therefore, is 13 times too large, and to obtain the correct product we must divide by 13. Dividing by 13, we have as the quotient, and as the answer to the exercise, $5\frac{7}{13}$.

Ex. 2. We are to multiply 12 by $\frac{3}{4}$.

12 multiplied by $\frac{1}{4}$ is $\frac{1}{4}$ of 12, or 3; and 12 multiplied by $\frac{3}{4}$ is 3 times 3, or 9.

Ex. 3. We are to multiply 10 by $\frac{7}{15}$.

10 multiplied by $\frac{1}{15}$ is $\frac{1}{15}$ of 10, or $\frac{10}{15}$, or $\frac{2}{3}$; and 10 multiplied by $\frac{7}{15}$ is 7 times $\frac{2}{3}$, or $\frac{14}{3}$, or $4\frac{2}{3}$.

Ex. 4. We are to multiply 144 by $\frac{49}{120}$.

We first think of 144 as multiplied by 49. But in multiplying by 49 integers instead of by $\frac{49}{120}$, we use a multiplier 120 times too large. Our product, therefore, is 120 times too large, and to obtain the correct product we must divide 144 times 49 by 120.

To shorten the operation, we mentally divide both dividend and divisor by 24, their greatest common factor.

We next divide the product of the remaining factors of the dividend by the remaining factor of the divisor, and obtain as the answer to the exercise $58\frac{4}{5}$.

Ex. 66.

Using the fraction as the multiplier, find the product of

1. $\frac{5}{8}$ and 7	11. $\frac{59}{49}$ and 72	21. $\frac{256}{829}$ and 244
2. $\frac{8}{9}$ and 6	12. $\frac{14}{25}$ and 18	22. $\frac{117}{450}$ and 163
3. $\frac{4}{7}$ and 6	13. $\frac{44}{79}$ and 31	23. $\frac{497}{523}$ and 361
4. $\frac{5}{9}$ and 5	14. $\frac{63}{91}$ and 37	24. $\frac{142}{780}$ and 540
5. $\frac{3}{4}$ and 7	15. $\frac{11}{26}$ and 64	25. $\frac{649}{701}$ and 695
6. $\frac{5}{6}$ and 3	16. $\frac{19}{20}$ and 80	26. $\frac{423}{607}$ and 143
7. $\frac{3}{8}$ and 6	17. $\frac{71}{80}$ and 45	27. $\frac{218}{150}$ and 799
8. $\frac{8}{9}$ and 5	18. $\frac{15}{99}$ and 33	28. $\frac{421}{780}$ and 469
9. $\frac{5}{7}$ and 8	19. $\frac{21}{50}$ and 75	29. $\frac{506}{970}$ and 710
10. $\frac{7}{8}$ and 4	20. $\frac{17}{80}$ and 63	30. $\frac{625}{763}$ and 148

52. To Multiply a Fraction by a Fraction.

We are to multiply $\frac{4}{5}$ by $\frac{7}{9}$.
We first multiply 4 by 7.
Explain the error arising from using 4 integers instead of $\frac{4}{5}$ as a multiplicand.
From using 7 integers instead of $\frac{7}{9}$ as a multiplier.

How may the effect of the first error be corrected?
Of the second error?
By what divisors, then, must we divide the product of 4 and 7?
Dividing by 5 and by 9 is equivalent to dividing by what one divisor?
How, then, in multiplying $\frac{4}{5}$ by $\frac{7}{9}$ do we find the divisor of the resulting fraction?
How did we find the dividend?
Give, then, a rule for multiplying a fraction by a fraction.

SOLUTIONS.

Ex. 1. Multiply $\frac{3}{5}$ by $\frac{7}{11}$.
Ex. 2. Multiply $\frac{9}{10}$ by $\frac{5}{6}$.
Ex. 3. Multiply $\frac{50}{63}$ by $\frac{21}{79}$.
Ex. 4. Multiply $\frac{27}{56}$ by $\frac{64}{135}$.

(1)	(2)	(3)	(4)
$\frac{21}{55}$	$\frac{3}{4}$	$\frac{50}{237}$	$\frac{\overset{1}{\cancel{27}} \times \overset{8}{\cancel{64}}}{\underset{7}{\cancel{56}} \times \underset{5}{\cancel{135}}} = \frac{8}{35}$

EXPLANATIONS.

Ex. 1. We are to multiply $\frac{3}{5}$ by $\frac{7}{11}$.

We first multiply 3 by 7. The product is 21. But in multiplying 3 integers by 7 integers we use a multiplicand 5 times too large and a multiplier 11 times too large. Our product, therefore, will be 11

times 5, or 55, times too large, and to obtain the correct product we must divide 21 by 55. The answer to the exercise, therefore, is 3 times 7 divided by 5 times 11, or $\frac{21}{55}$.

Ex. 2. Explain the solution of Ex. 2.

Ex. 3. We are to multiply $\frac{50}{63}$ by $\frac{21}{79}$.

We first think of 50 and 21 as factors of the final dividend. But in multiplying 50 integers by 21 integers we should use a multiplicand 63 times too large and a multiplier 79 times too large, and to obtain the correct product it would be necessary to divide 50 times 21 by 63 times 79.

The factor 21 is common to both dividend and divisor. We therefore mentally cancel this factor before performing the multiplications indicated. Multiplying together the remaining factors of the dividend and of the divisor, we have as the answer to the exercise $50 \div 237$, or $\frac{50}{237}$.

Ex. 4. We are to multiply $\frac{27}{56}$ by $\frac{64}{135}$.

We first indicate the multiplication of 27 by 64. But in multiplying 27 integers by 64 integers we use a multiplicand 56 times too large and a multiplier 135 times too large. Our product, therefore, is 56 times 135 times too large, and to obtain the correct product we must divide 27 times 64 by 56 times 135.

To shorten the work we cancel the factors common to the dividend and the divisor. We next multiply together the remaining factors of the dividend and of the divisor, and obtain as the result of the exercise $\frac{8}{35}$.

Ex. 67.

Multiply

1. $\frac{3}{4}$ by $\frac{2}{5}$	11. $\frac{10}{11}$ by $\frac{22}{30}$	21. $\frac{162}{426}$ by $\frac{214}{405}$
2. $\frac{5}{7}$ by $\frac{3}{2}$	12. $\frac{17}{18}$ by $\frac{16}{21}$	22. $\frac{110}{625}$ by $\frac{125}{320}$
3. $\frac{6}{7}$ by $\frac{7}{9}$	13. $\frac{11}{20}$ by $\frac{18}{44}$	23. $\frac{155}{193}$ by $\frac{601}{185}$
4. $\frac{2}{7}$ by $\frac{8}{9}$	14. $\frac{13}{16}$ by $\frac{26}{11}$	24. $\frac{675}{722}$ by $\frac{118}{220}$
5. $\frac{4}{5}$ by $\frac{3}{5}$	15. $\frac{23}{24}$ by $\frac{47}{82}$	25. $\frac{190}{816}$ by $\frac{540}{250}$
6. $\frac{8}{9}$ by $\frac{5}{6}$	16. $\frac{17}{27}$ by $\frac{43}{81}$	26. $\frac{120}{890}$ by $\frac{135}{144}$
7. $\frac{7}{8}$ by $\frac{8}{9}$	17. $\frac{19}{40}$ by $\frac{62}{90}$	27. $\frac{216}{519}$ by $\frac{820}{144}$
8. $\frac{5}{6}$ by $\frac{7}{9}$	18. $\frac{37}{41}$ by $\frac{23}{74}$	28. $\frac{450}{112}$ by $\frac{265}{540}$
9. $\frac{4}{9}$ by $\frac{2}{5}$	19. $\frac{34}{37}$ by $\frac{22}{56}$	29. $\frac{703}{720}$ by $\frac{360}{370}$
10. $\frac{6}{7}$ by $\frac{7}{8}$	20. $\frac{43}{90}$ by $\frac{25}{37}$	30. $\frac{550}{633}$ by $\frac{464}{550}$

53. To Multiply when the Multiplier or the Multiplicand is a Mixed Number.

$4\frac{2}{3}+35=39\frac{2}{3}$

We are to multiply $5\frac{2}{3}$ by 7.
What sign is understood between the 5 and the $\frac{2}{3}$?
$\frac{2}{3}$ multiplied by 7 gives what product?
5 multiplied by 7 gives what product?
What do we do with these partial products?
Give, then, a rule for multiplying a mixed number by an integer.

We are to multiply 7 by $5\frac{2}{3}$.
What sign is understood between the 5 and the $\frac{2}{3}$?
7 multiplied by $\frac{2}{3}$ gives what product?
7 multipled by 5 gives what product?
What do we do with these partial products?
Give, then, a rule for multiplying an integer by a mixed number.

Ex. 1. Multiply $5\frac{1}{7}$ by 4.
Ex. 2. Multiply $7\frac{3}{4}$ by 5.
Ex. 3. Multiply $324\frac{15}{17}$ by 25.

SOLUTIONS.

(1)	(2)	(3)
$20\frac{4}{7}$	$38\frac{3}{4}$	375
		$22\frac{1}{17}$
		8100
		$8122\frac{1}{17}$

EXPLANATIONS.

Ex. 1. We are to multiply $5\frac{1}{7}$ by 4.

$5\frac{1}{7}$ equals 5 plus $\frac{1}{7}$. 4 times 5 are 20, and 4 times $\frac{1}{7}$ are $\frac{4}{7}$. 20 plus $\frac{4}{7}$ equals $20\frac{4}{7}$.

Ex. 2. We are to multiply $7\frac{3}{4}$ by 5.

$7\frac{3}{4}$ equals 7 plus $\frac{3}{4}$. 5 times 7 are 35, and 5 times $\frac{3}{4}$ equals $\frac{15}{4}$, or $3\frac{3}{4}$. 35 plus $3\frac{3}{4}$ equals $38\frac{3}{4}$. Therefore the product of $7\frac{3}{4}$ by 5 is $38\frac{3}{4}$.

Ex. 3. Explain the solution of Ex. 3.

Ex. 1. Multiply 24 by $3\frac{5}{6}$.
Ex. 2. Multiply 46 by $5\frac{3}{7}$.
Ex. 3. Multiply 324 by $37\frac{3}{8}$.
Ex. 4. Multiply 8244 by $568\frac{7}{9}$.

SOLUTIONS.

(1)	(2)	(3)	(4)
92	138	972	916
	$19\frac{5}{7}$	$121\frac{1}{2}$	6412
	230	2268	65952
	$249\frac{5}{7}$	972	461664
		$12109\frac{1}{2}$	4689004

EXPLANATIONS.

Ex. 1. We are to multiply 24 by $3\frac{5}{6}$.

$3\frac{5}{6}=4-\frac{1}{6}$. $24\times 4=96$, and $24\times\frac{1}{6}=4$. $96-4=92$. Therefore, $24\times 3\frac{5}{6}=96-4$, or 92.

Ex. 2. We are to multiply 46 by $5\frac{3}{7}$.

$5\frac{3}{7}$ equals 5 plus $\frac{3}{7}$. The product of 46 by $\frac{3}{7}$ is $19\frac{5}{7}$, and the prod-

uct of 46 by 5 is 230. $19\frac{5}{7}$ plus 230 equals $249\frac{5}{7}$. Therefore the product of 46 by $5\frac{3}{7}$ is $249\frac{5}{7}$.

Ex. 3. We are to multiply 324 by $37\frac{3}{8}$.

$37\frac{3}{8}$ equals 3 tens and 7 units plus 3 eighths. $324 \times \frac{3}{8} = 112\frac{1}{2}$, $324 \times 7 = 2268$, and 324×3 tens $= 972$ tens. The sum of these three partial products is $12109\frac{1}{2}$. Therefore, the product of 324 by $37\frac{3}{8}$ is $12109\frac{1}{2}$.

Ex. 4. Explain the solution of Ex. 4.

NOTE. To avoid confusion in the relative position of the partial products find first that partial product of which the fractional part of the multiplicand is one of the factors. Thus, to multiply 4842 by $768\frac{5}{8}$, first multiply by $\frac{5}{8}$.

If, however, the numerator of the fractional part of the multiplier is one of the figures of the integral part, multiply first by the integral part, commencing with its left-hand figure.

Thus, to multiply by $657\frac{5}{8}$, multiply by the 6, by the 5, and by the 7; then divide the product by the 5 by 8.

What labor is saved by this process ?

If a figure in the integral part of the multiplier is a multiple of the numerator of the fractional part, the same artifice may be employed.

Thus, to multiply by a number like $569\frac{2}{3}$, multiply by the 5, 6, and 9; then, thinking of the $\frac{2}{3}$ as $\frac{6}{9}$, complete the multiplication as directed in the preceding paragraphs.

To multiply by a number like $5\frac{6}{7}$ increase the integer by 1 and make the necessary correction.

Give a rule for multiplying, according to this method, by $15\frac{3}{4}$; by $19\frac{4}{5}$; by $43\frac{34}{35}$; by $99\frac{3}{7}$.

Ex. 68.

Multiply

1. $7\frac{3}{4}$ by 5	11. 25 by $17\frac{24}{25}$	21. $215\frac{336}{86}$ by 125
2. $8\frac{2}{3}$ by 7	12. 24 by $34\frac{43}{56}$	22. $324\frac{125}{250}$ by 324
3. $9\frac{5}{8}$ by 4	13. 82 by $75\frac{37}{40}$	23. $795\frac{327}{434}$ by 196
4. $7\frac{1}{2}$ by 8	14. 47 by $95\frac{75}{79}$	24. $835\frac{273}{319}$ by 224
5. $8\frac{3}{8}$ by 7	15. 34 by $52\frac{25}{68}$	25. $133\frac{355}{409}$ by 201
6. $5\frac{2}{3}$ by 4	16. 64 by $78\frac{29}{32}$	26. $127\frac{370}{980}$ by 726
7. $4\frac{5}{8}$ by 7	17. 44 by $32\frac{27}{33}$	37. $327\frac{234}{300}$ by 939.
8. $6\frac{1}{4}$ by 2	18. 65 by $14\frac{45}{64}$	28. $535\frac{223}{334}$ by 734
9. $5\frac{7}{9}$ by 4	19. 36 by $39\frac{37}{48}$	29. $139\frac{113}{224}$ by 420
10. $9\frac{5}{6}$ by 3	20. 59 by $39\frac{25}{42}$	30. $539\frac{105}{119}$ by 187

54. To Multiply a Mixed Number by a Mixed Number.

We are to multiply $2\frac{3}{4}$ by $3\frac{1}{3}$.

$\frac{11}{4} \times \frac{10}{3} = \frac{110}{12} = 9\frac{1}{6}$

$2\frac{3}{4} = ?$
$3\frac{1}{3} = ?$
$\frac{11}{4} \times \frac{10}{3} = ?$
$2\frac{3}{4} \times 3\frac{1}{3}$, then, $= ?$
Give, then, a rule for multiplying a mixed number by a mixed number.

We are to multiply $97\frac{1}{5}$ by $80\frac{1}{9}$.
What sign is understood between the integral and the fractional part of the multiplicand and of the multiplier?

$$\begin{array}{r} 7760 \\ 16 \\ 10\frac{7}{9} \\ \frac{1}{45} \\ \hline 7786\frac{4}{5} \end{array}$$

We may, then, think of our multiplicand as consisting of how many parts?
Of our multiplier?

What parts do we first multiply together?
What product do we obtain?

What are the second parts that we multiply together?
What product do we obtain?

What are the third parts that we multiply together?
What product do we obtain?

What are the fourth parts that we multiply together?
What product do we obtain?
What, then, is our total product?
Give, then, a special rule for multiplication of mixed numbers when the numerator of each fractional part is 1.

* * *

We are to multiply $29\frac{2}{5}$ by $29\frac{3}{5}$. $870\frac{6}{25}$

We may think of our multiplicand as consisting of how many parts?
Of our multiplier?

What are the first parts to be multiplied together?
The second parts?
The third parts?
The fourth parts?

29 multiplied by $\frac{2}{5}$ plus 29 multiplied by $\frac{3}{5}$ equals 29 multiplied by what?
29 multiplied by 29 plus 29 multiplied by 1 equals 29 multiplied by what?
What one product, then, is equivalent to the first three partial products?

$29 \times 30 = ?$
$\frac{2}{5} \times \frac{3}{5} = ?$
$29\frac{2}{5} \times 29\frac{3}{5}$, then, $= ?$
Give, then, a special rule for multiplication of mixed num-

bers when the integral parts of the two factors are the same, and the sum of the fractional parts is 1.

Solve mentally and explain the solution of each of the following exercises:

Multiply

1. $7\frac{1}{2}$ by $7\frac{1}{2}$	6. $9\frac{3}{4}$ by $9\frac{1}{4}$	11. $30\frac{1}{5}$ by $40\frac{1}{3}$
2. $9\frac{1}{2}$ by $9\frac{1}{2}$	7. $10\frac{3}{7}$ by $10\frac{4}{7}$	12. $8\frac{1}{5}$ by $15\frac{1}{4}$
3. $11\frac{1}{2}$ by $11\frac{1}{2}$	8. $8\frac{5}{11}$ by $8\frac{6}{11}$	13. $12\frac{1}{7}$ by $14\frac{1}{6}$
4. $49\frac{1}{2}$ by $49\frac{1}{2}$	9. $11\frac{4}{13}$ by $11\frac{9}{13}$	14. $9\frac{1}{5}$ by $40\frac{1}{3}$
5. $99\frac{1}{2}$ by $99\frac{1}{2}$	10. $19\frac{7}{15}$ by $19\frac{8}{15}$	15. $10\frac{3}{8}$ by $16\frac{2}{5}$

Ex. 1. Multiply $11\frac{4}{15}$ by $5\frac{5}{26}$.

Ex. 2. Multiply $29\frac{1}{2}$ by $29\frac{1}{2}$.

Ex. 3. Multiply $75\frac{7}{12}$ by $40\frac{11}{12}$.

SOLUTIONS.

(1)

$$\frac{\overset{13}{\cancel{169}} \times \overset{9}{\cancel{135}}}{\cancel{15} \times \underset{2}{\cancel{26}}} = \frac{117}{2} = 58\frac{1}{2}$$

(2)

$870\frac{1}{4}$

(3)

$$\begin{array}{r} 3000 \\ 92\frac{1}{12} \\ \frac{77}{144} \\ \hline 3092\frac{89}{144} \end{array} \qquad \begin{array}{r} 280 \\ 825 \\ \hline 1105 \end{array}$$

EXPLANATIONS.

Ex. 1. We are to multiply $11\frac{4}{15}$ by $5\frac{5}{26}$.

$11\frac{4}{15} = \frac{169}{15}$ and $5\frac{5}{26} = \frac{135}{26}$. Therefore, $11\frac{4}{15} \times 5\frac{5}{26} = \frac{169}{15} \times \frac{135}{26}$, or $58\frac{1}{2}$.

Ex 2. We are to multiply $29\frac{1}{2}$ by $29\frac{1}{2}$.

$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. 29 multiplied by $\frac{1}{2}$ plus $\frac{1}{2}$ multiplied by 29 equals once 29. Once 29 plus 29 times 29 equals 30 times 29, or 870. Therefore, $29\frac{1}{2} \times 29\frac{1}{2} = (29 \times 30) + (\frac{1}{2} \times \frac{1}{2})$, or $870\frac{1}{4}$.

Ex. 3. We are to multiply $75\frac{7}{12}$ by $40\frac{11}{12}$,

$75 \times 40 = 3000$, $\frac{7}{12} \times 40 = \frac{280}{12}$, and $75 \times \frac{11}{12} = \frac{825}{12}$. $\frac{280}{12} + \frac{825}{12} = \frac{1105}{12}$, or $92\frac{1}{12}$. $\frac{7}{12} \times \frac{11}{12} = \frac{77}{144}$. Therefore, $75\frac{7}{12} \times 40\frac{11}{12} = 3000 + 92\frac{1}{12} + \frac{77}{144}$, or $3092\frac{89}{144}$.

NOTE I. The process used in Ex. 3 may be profitably followed whenever the integral parts of the mixed number are large and the fractional parts are comparatively small and have the same denominator.

Keep this process in mind in solving those problems in mensuration in which the dimensions are expressed in feet and inches.

NOTE. 2. Suppose that we are to multiply $46\frac{5}{6}$ by $46\frac{1}{12}$.

46 multiplied by $\frac{5}{6}$ plus 46 multiplied by $\frac{1}{12}$ equals 46 multiplied by what ?

Give a general rule applying to exercises of this class.

Suppose that we are to multiply $37\frac{5}{12}$ by $47\frac{2}{3}$, What change may be made in the fractional part of the multiplier that will enable us to employ the process used in the third solution on the preceding page?

Ex. 69.

Multiply

1. $5\frac{2}{3}$ by $7\frac{3}{5}$	11. $18\frac{3}{5}$ by $13\frac{4}{7}$	21. $183\frac{3}{5}$ by $275\frac{2}{3}$
2. $8\frac{3}{7}$ by $5\frac{4}{5}$	12. $81\frac{3}{5}$ by $20\frac{5}{6}$	22. $476\frac{5}{6}$ by $493\frac{2}{3}$
3. $6\frac{5}{9}$ by $4\frac{2}{3}$	13. $79\frac{1}{9}$ by $83\frac{1}{7}$	23. $816\frac{1}{2}$ by $578\frac{3}{4}$
4. $7\frac{1}{2}$ by $8\frac{1}{2}$	14. $35\frac{4}{9}$ by $35\frac{5}{9}$	24. $735\frac{5}{7}$ by $735\frac{2}{7}$
5. $7\frac{1}{2}$ by $7\frac{1}{2}$	15. $47\frac{1}{2}$ by $47\frac{1}{2}$	25. $235\frac{17}{20}$ by $947\frac{23}{25}$
6. $8\frac{1}{2}$ by $8\frac{1}{2}$	16. $29\frac{13}{15}$ by $54\frac{17}{19}$	26. $438\frac{11}{12}$ by $673\frac{5}{12}$
7. $9\frac{4}{7}$ by $9\frac{5}{9}$	17. $73\frac{5}{12}$ by $91\frac{5}{12}$	27. $892\frac{7}{12}$ by $415\frac{5}{16}$
8. $4\frac{2}{3}$ by $3\frac{5}{9}$	18. $62\frac{23}{25}$ by $78\frac{39}{47}$	28. $316\frac{113}{273}$ by $374\frac{213}{265}$
9. $4\frac{2}{3}$ by $2\frac{3}{7}$	19. $27\frac{13}{16}$ by $87\frac{11}{16}$	29. $873\frac{113}{257}$ by $873\frac{144}{257}$
10. $7\frac{3}{4}$ by $1\frac{1}{2}$	20. $58\frac{7}{11}$ by $44\frac{5}{29}$	30. $675\frac{123}{144}$ by $796\frac{137}{144}$

55. To Divide a Fraction by an Integer.

We are to find the value of the expression $\frac{24}{25} \div 4$.
How many and what signs of division do we have in this expression?
Which number, then, is a dividend?
Which numbers are divisors?
By which divisor alone is 24 divisible?
What, then, will be the first step of the solution?

$24 \div 4 = ?$
How may the division of 6 by 25 be indicated?
What, then, is the value of the expression?
Give, then, a rule for dividing a fraction by an integer when the integer is contained in the dividend of the fraction.

We are to find the value of the expression $\frac{23}{25} \div 4$.
Which number in the expression is the dividend?
Which numbers are divisors?
Is the dividend divisible by either divisor?
Dividing by 5 and by 25 is equivalent to dividing by what one divisor?
How, then, can we find a single divisor equivalent to the divisors 25 and 4?

25×4 gives what product?
What, then, is the final divisor?
What was the dividend?
What, then, is the value of the expression?

Give, then, a rule for dividing a fraction by an integer when the integer is not contained in the dividend of the fraction.

		SOLUTIONS.			
Ex. 1.	**Divide $\frac{24}{25}$ by 6.**	(1)	(2)	(3)	(4)
Ex. 2.	**Divide $\frac{7}{8}$ by 9.**	$\frac{4}{25}$	$\frac{7}{72}$	$\frac{2}{27}$	$\frac{3}{116}$
Ex. 3.	**Divide $\frac{8}{9}$ by 12.**				
Ex. 4.	**Divide $\frac{21}{29}$ by 28.**	**EXPLANATIONS.**			

Ex. 1. We are to divide $\frac{24}{25}$ by 6.

In this exercise 24 is a dividend, and 25 and 6 are divisors. We first divide 24 by the divisor 6. The quotient is 4. 4 is not divisible by 25, but its division may be indicated. Indicating the division, we have as the answer to the exercise $\frac{4}{25}$.

Ex. 2. We are to divide $\frac{7}{8}$ by 9.

In this exercise 7 is a dividend and 8 and 9 are divisors. 7 is divisible by neither divisor. Our dividend, therefore, remains unchanged, while we obtain a new divisor by multiplying together the divisors 8 and 9. The answer to the problem, therefore, is 7 divided by 8 times 9, or $\frac{7}{72}$.

Ex. 3. We are to divide $\frac{8}{9}$ by 12.

In this exercise 8 is a dividend, and 9 and 12 are divisors. 8 is divisible by neither divisor. The dividend 8 and the divisor 12 contain, however, the common factor 4. Mentally removing this factor, we have as our final dividend 2, and as our final divisor 3 times 9, or 27. The answer to the exercise, therefore, is 2 divided by 27, or $\frac{2}{27}$.

Ex. 4. Explain the solution of Ex. 4.

Ex. 70.

Find the quotient of

1. $\frac{3}{8} \div 2$	**11.** $\frac{75}{92} \div 25$	**21.** $\frac{245}{734} \div 350$
2. $\frac{2}{3} \div 2$	**12.** $\frac{74}{83} \div 32$	**22.** $\frac{896}{775} \div 448$
3. $\frac{8}{9} \div 4$	**13.** $\frac{28}{37} \div 21$	**23.** $\frac{279}{435} \div 297$
4. $\frac{5}{6} \div 4$	**14.** $\frac{64}{70} \div 15$	**24.** $\frac{625}{737} \div 250$
5. $\frac{6}{7} \div 3$	**15.** $\frac{95}{99} \div 19$	**25.** $\frac{326}{437} \div 236$
6. $\frac{5}{8} \div 7$	**16.** $\frac{68}{75} \div 51$	**26.** $\frac{360}{428} \div 216$
7. $\frac{4}{7} \div 6$	**17.** $\frac{69}{70} \div 26$	**27.** $\frac{307}{634} \div 699$
8. $\frac{3}{5} \div 5$	**18.** $\frac{50}{63} \div 25$	**28.** $\frac{536}{834} \div 125$
9. $\frac{7}{8} \div 3$	**19.** $\frac{84}{92} \div 24$	**29.** $\frac{350}{417} \div 250$
10. $\frac{8}{9} \div 4$	**20.** $\frac{87}{98} \div 58$	**30.** $\frac{270}{729} \div 135$

56. To Divide a Fraction or an Integer by a Fraction.

We are to find the value of the expression $\frac{2}{3} \div \frac{5}{7}$.

In the expression $\frac{2}{3} \div 5$, which number is a dividend? Which numbers are divisors?

Compare the two expressions. How does the part of the first expression preceding the sign $\div$ compare with the corresponding part of the second expression?

How does the part of the first expression following the sign $\div$ compare with the corresponding part of the second expression?

Dividing a divisor has what effect upon the value of an expression of division.

The 7, then, being a divisor of a divisor, may be thought of as what element of the entire expression?

What, then, are the dividends of the first expression?

What are the divisors?

Multiplying the dividend 2 by the dividend 7 gives what final dividend?

Multiplying the divisor 3 by the divisor 5 gives what final divisor?

What, then, is the value of the expression?

In dividing a fraction by a fraction the final dividend is always the product of two factors.

The first factor is what element of the expression preceding the sign $\div$?

The second factor is what element of the expression following the sign $\div$?

The final divisor is also always the product of two factors.

The first factor is what element of the expression preceding the sign $\div$?

The second factor is what element of the expression following the sign $\div$?

Give, then, a rule for dividing a fraction by a fraction.

How will the quotient compare with the dividend when the divisor is a proper fraction?

How when the divisor is an improper fraction?

Name, without written work, the value of

$\frac{1}{2} \div \frac{2}{3}$ $\frac{2}{5} \div \frac{3}{7}$ $\frac{4}{9} \div \frac{7}{8}$ $\frac{3}{5} \div \frac{2}{7}$ $\frac{3}{7} \div \frac{5}{8}$ $\frac{4}{7} \div \frac{3}{5}$ $\frac{5}{9} \div \frac{7}{8}$

$\frac{15}{17} \div \frac{7}{9}$ $\frac{13}{18} \div \frac{6}{7}$ $\frac{23}{37} \div \frac{2}{3}$ $\frac{17}{19} \div \frac{5}{8}$ $\frac{13}{88} \div \frac{7}{8}$ $\frac{23}{37} \div \frac{3}{4}$ $\frac{17}{18} \div \frac{8}{9}$

We are to find the value of the expression $\frac{8}{15} \div \frac{24}{25}$.

In this expression which numbers are dividends?

Which numbers are divisors?

What factor is common to the dividend 8 and the divisor 24?

Removing this common factor, what dividend remains in place of the dividend 8?

What divisor in place of the divisor 24?

What factor is common to the divisor 15 and the dividend 25?

Removing this common factor, what divisor remains in place of the divisor 15?

What dividend in place of the dividend 25?

What, then, is the final dividend?

What is the final divisor?

What is the value of the expression?

Give a rule for dividing a fraction by a fraction when there are factors common to a dividend and a divisor.

Name the values of the following expressions:

$\frac{14}{85} \div \frac{21}{25}$	$\frac{16}{27} \div \frac{28}{45}$	$\frac{22}{35} \div \frac{33}{40}$	$\frac{26}{51} \div \frac{39}{68}$	$\frac{10}{21} \div \frac{25}{28}$
$\frac{23}{25} \div \frac{69}{75}$	$\frac{45}{51} \div \frac{25}{34}$	$\frac{12}{21} \div \frac{16}{63}$	$\frac{49}{64} \div \frac{77}{96}$	$\frac{26}{45} \div \frac{78}{90}$

* * *

We are to find the value of the expression $3 \div \frac{4}{5}$.

The 3 of the expression is what element? the 4? the 5?

Is either dividend divisible by the divisor?

What, then, is the final dividend?

What is the final divisor?

What is the value of the expression?

Give a rule for dividing an integer by a fraction when the dividend of the fraction is not contained in the integer.

We are to find the value of the expression $6 \div \frac{3}{5}$.

Which numbers are dividends in this expression?

Which number is a divisor?

Is either dividend divisible by the divisor?

$6 \div 3$ gives what quotient?

What, then, are our final dividends?

What is the value of the expression?

Give a rule for dividing an integer by a fraction when the dividend of the fraction is contained in the integer.

We are to find the value of the expression $25 \div \frac{10}{13}$.

In this expression which numbers are dividends?

Which number is a divisor?

Is either dividend divisible by the divisor?

What factor is common to the dividend 25 and the divisor 10?

Removing this common factor what dividend have we in place of the dividend 25?

What divisor in place of the divisor 10?

What, then, is the final dividend?

What is the final divisor?

What is the value of the expression?

Give a rule for dividing an integer by a fraction when there are factors common to the integer and the dividend of the fraction?

Name, without written work, the value of

$3 \div \frac{5}{7}$	$6 \div \frac{5}{9}$	$8 \div \frac{3}{4}$	$2 \div \frac{7}{9}$	$8 \div \frac{3}{7}$	$4 \div \frac{7}{8}$	$7 \div \frac{5}{8}$
$2 \div \frac{3}{7}$	$9 \div \frac{4}{5}$	$6 \div \frac{5}{8}$	$9 \div \frac{4}{7}$	$7 \div \frac{2}{9}$	$8 \div \frac{3}{5}$	$9 \div \frac{4}{5}$
$6 \div \frac{2}{7}$	$6 \div \frac{3}{7}$	$8 \div \frac{4}{5}$	$6 \div \frac{2}{9}$	$8 \div \frac{2}{3}$	$9 \div \frac{3}{5}$	$8 \div \frac{4}{9}$
$10 \div \frac{5}{9}$	$21 \div \frac{3}{4}$	$48 \div \frac{4}{5}$	$35 \div \frac{7}{8}$	$32 \div \frac{4}{7}$	$36 \div \frac{2}{7}$	$42 \div \frac{7}{8}$

$15 \div \frac{20}{21}$	$24 \div \frac{18}{19}$	$12 \div \frac{15}{17}$	$64 \div \frac{48}{49}$	$75 \div \frac{50}{53}$	$36 \div \frac{28}{31}$
$42 \div \frac{35}{43}$	$33 \div \frac{44}{45}$	$49 \div \frac{14}{37}$	$27 \div \frac{18}{25}$	$35 \div \frac{14}{19}$	$60 \div \frac{18}{23}$

SOLUTIONS.

Ex. 1. Divide 24 by $\frac{8}{9}$.
Ex. 2. Divide 11 by $\frac{8}{9}$.
Ex. 3. Divide 75 by $\frac{50}{62}$.
Ex. 4. Divide 6160 by $\frac{264}{317}$.

(1) 27 (2) $12\frac{3}{8}$ (3) 93

(4)
$$\overset{70}{\cancel{\overset{560}{\cancel{6160}}}} \div \frac{\overset{3}{\cancel{\overset{24}{\cancel{264}}}}}{317} = \frac{22190}{3}$$
$$= 7396\frac{2}{3}$$

EXPLANATIONS.

Ex. 1. We are to divide 24 by $\frac{8}{9}$.

In this exercise 24 is a dividend, 8 is a divisor, and 9 a divisor of a divisor, or a dividend.

24 divided by 8 equals 3. Our final dividend, therefore, is 3 times 9, or 27.

Ex. 2. We are to divide 11 by $\frac{8}{9}$.

In this exercise 11 is a dividend, 8 is a divisor, and 9 is a divisor of a divisor, or a dividend.

Neither dividend is divisible by the divisor. The final dividend, therefore, is 9 times 11, or 99, and the answer to the exercise is 99 divided by 8, or $12\frac{3}{8}$.

Ex. 3. We are to divide 75 by $\frac{50}{62}$.

In this exercise 75 is a dividend, 50 is a divisor, and 62 is a divisor of a divisor, or a dividend.

Neither dividend is divisible by the divisor. We can, however, cancel the factors 25 and 2 from the divisor 50 and the dividends 75 and 62.

Mentally cancelling these factors, we have as the final dividend, and as the value of the expression, 3 times 31, or 93.

Ex. 4. Explain the solution of Ex. 4.

Ex. 71.

Divide

1. 35 by $\frac{12}{23}$
2. 46 by $\frac{37}{39}$
3. 84 by $\frac{48}{65}$
4. 75 by $\frac{50}{81}$
5. 94 by $\frac{28}{47}$
6. 63 by $\frac{72}{96}$
7. 27 by $\frac{38}{43}$
8. 55 by $\frac{44}{47}$
9. 72 by $\frac{48}{57}$
10. 75 by $\frac{25}{29}$
11. 144 by $\frac{24}{37}$
12. 143 by $\frac{44}{57}$
13. 996 by $\frac{34}{45}$
14. 324 by $\frac{45}{47}$
15. 816 by $\frac{45}{53}$
16. 933 by $\frac{23}{25}$
17. 352 by $\frac{75}{84}$
18. 248 by $\frac{39}{57}$
19. 721 by $\frac{42}{65}$
20. 500 by $\frac{25}{99}$
21. 153 by $\frac{34}{45}$
22. 325 by $\frac{65}{82}$
23. 484 by $\frac{64}{75}$
24. 347 by $\frac{67}{85}$
25. 936 by $\frac{72}{99}$
26. 594 by $\frac{88}{93}$
27. 648 by $\frac{64}{85}$
28. 536 by $\frac{48}{53}$
29. 575 by $\frac{75}{78}$

20. 428 by $\frac{25}{37}$

SOLUTIONS.

Ex. 1. Divide $\frac{3}{8}$ by $\frac{7}{11}$.
Ex. 2. Divide $\frac{16}{25}$ by $\frac{24}{35}$.
Ex. 3. Divide $\frac{56}{135}$ by $\frac{48}{90}$.
Ex. 4. Divide $1\frac{31}{144}$ by $1\frac{29}{96}$.
Ex. 5. Divide 2837 by $30\frac{1}{4}$.

(1) $\frac{33}{56}$

(2) $\frac{14}{15}$

(3) $\frac{56}{135} \div \frac{8}{15} = \frac{7}{9}$

(4)
$$\frac{\overset{7}{175}}{\underset{\underset{3}{12}}{144}} \div \frac{\overset{5}{125}}{\underset{\underset{2}{8}}{96}} = \frac{14}{15}$$

(5)
$$\begin{array}{l} 11348 \div 121 \\ \underline{1089} \quad = 93\frac{95}{121} \\ 458 \\ 363 \end{array}$$

EXPLANATIONS.

Ex. 1. We are to divide $\frac{3}{8}$ by $\frac{7}{11}$.

In this exercise 3 is a dividend, 8 is a divisor, 7 is a divisor, and 11 is a divisor of a divisor, or a dividend. Our final dividend, therefore, is 3 times 11, or 33, and our final divisor 7 times 8, or 56.

The answer to the exercise, therefore, is 33 divided by 56, or $\frac{33}{56}$.

Ex. 2. We are to divide $\frac{16}{25}$ by $\frac{24}{35}$.

In this exercise 16 is a dividend, 25 is a divisor, 24 is a divisor, and 35 is a divisor of a divisor, or a dividend.

The dividend 16 and the divisor 24 have the common factor 8, and the divisor 25 and the dividend 35 the common factor 5. Mentally removing these common factors, we have as our final dividends 2 and 7, and as our final divisors 3 and 5. The answer to the exercise, therefore, is 2 times 7 divided by 3 times 5, or 14 divided by 15, or $\frac{14}{15}$.

Ex. 3. Explain the solution of Ex. 3.

Ex. 4. Explain the solution of Ex 4.

Ex. 5. Explain the solution of Ex. 5.

NOTE. Observe that in Ex. 5 we multiply both dividend and divisor by a number which produces an integral divisor. A similar process might have been employed in Ex. 4; that is, as the first step of the solution, both fractions might have been multiplied by 288, the least common multiple of 144 and 96.

Solve Ex. 4 by this method. Compare the process, step by step, with that used in the book, and decide which solution you prefer.

Ex. 72.

Divide

1. $\frac{27}{35}$ by $\frac{40}{63}$	11. $12\frac{3}{5}$ by $23\frac{4}{9}$	21. 417 by $5\frac{2}{3}$
2. $\frac{84}{95}$ by $\frac{26}{38}$	12. $37\frac{3}{8}$ by $14\frac{2}{3}$	22. 958 by $8\frac{3}{7}$
3. $\frac{74}{93}$ by $\frac{34}{37}$	13. $23\frac{7}{9}$ by $19\frac{7}{8}$	23. 475 by $6\frac{3}{5}$
4. $\frac{66}{77}$ by $\frac{48}{99}$	14. $32\frac{3}{5}$ by $57\frac{4}{7}$	24. 763 by $23\frac{7}{12}$
5. $\frac{76}{85}$ by $\frac{36}{68}$	15. $84\frac{7}{9}$ by $27\frac{5}{8}$	25. 968 by $49\frac{17}{40}$
6. $\frac{47}{59}$ by $\frac{67}{85}$	16. $35\frac{13}{17}$ by $24\frac{7}{11}$	26. 486 by $59\frac{13}{15}$
7. $\frac{69}{73}$ by $\frac{92}{97}$	17. $58\frac{39}{76}$ by $24\frac{17}{57}$	27. 658 by $34\frac{23}{25}$
8. $\frac{57}{64}$ by $\frac{84}{96}$	18. $31\frac{17}{40}$ by $84\frac{31}{60}$	28. 542 by $75\frac{83}{125}$
9. $\frac{58}{67}$ by $\frac{87}{89}$	19. $29\frac{13}{37}$ by $56\frac{43}{59}$	29. 947 by $84\frac{73}{99}$
10. $\frac{54}{73}$ by $\frac{39}{52}$	20. $84\frac{17}{50}$ by $43\frac{49}{75}$	30. 526 by $47\frac{123}{370}$

57. Compound and Complex Fractions.

The name **Compound Fraction** is applied to an expression consisting of two or more fractions connected by the word "of," and the name **Complex Fraction** to a fraction having a fraction as or in one or both of its terms.

The process of simplifying a complex fraction consists simply in performing the indicated operations. In simplifying a compound fraction remember that "$\frac{3}{4}$ of $\frac{5}{7}$," for example, is equivalent to "3 times $\frac{1}{4}$ of $\frac{5}{7}$." 4, therefore, is a factor of the final divisor, and 3 a factor of the final dividend, and the expression is equivalent to and interchangeable with $\frac{3}{4} \times \frac{5}{7}$.

NOTE. Write the expression $\frac{5}{7} \div \frac{9}{11}$ in the form of a complex fraction.

The 5 is what element in the expression thus produced? the 7? the 9; the 11; How, then, do we simplify the fraction.

Express the first ten exercises in Ex. 72 in the form of complex fractions, and find their value. Give a rule for simplifying complex fractions of this form.

Ex. 73.

Simplify the following expressions:

1. $\dfrac{\frac{3}{4}+\frac{5}{8}}{\frac{3}{4}-\frac{2}{3}}$	5. $\dfrac{\frac{23}{30}\times\frac{45}{46}}{\frac{40}{51}\div\frac{25}{34}}$	9. $\dfrac{\frac{7}{8}\text{ of }7\frac{7}{8}}{3\frac{3}{4}\times 5\frac{2}{3}}$
2. $\dfrac{\frac{7}{8}\times\frac{1}{3}}{\frac{1}{4}\times\frac{3}{8}}$	6. $\dfrac{\frac{56}{75}\div\frac{25}{34}}{\frac{38}{51}-\frac{34}{57}}$	10. $\dfrac{\frac{7}{8}\text{ of }4\frac{4}{7}}{\frac{1}{8}\div 8\frac{1}{2}}$
3. $\dfrac{\frac{1}{4}\text{ of }\frac{3}{8}}{\frac{1}{2}\text{ of }\frac{1}{4}}$	7. $\dfrac{\frac{49}{69}\times\frac{46}{63}}{\frac{28}{35}+\frac{27}{45}}$	11. $\dfrac{4\frac{1}{4}-\frac{1}{2}}{\frac{3}{8}\div 6\frac{1}{3}}$
4. $\dfrac{\frac{1}{2}-\frac{1}{8}}{\frac{1}{4}+\frac{1}{8}}$	8. $\dfrac{\frac{23}{24}\times\frac{24}{23}}{\frac{39}{47}\div\frac{78}{94}}$	12. $\dfrac{5\frac{3}{8}\div\frac{7}{8}}{\frac{7}{8}\div 5\frac{3}{8}}$

58. To Change a Common Fraction to an Equivalent Decimal.

We are to change $\frac{3}{4}$ to an equivalent decimal. To do this, we will reduce the 3 units to some lower order, and then perform the indicated division.

By what do we multiply a number of any order to reduce it to the next lower order?

What are the prime factors of 4?

Imagine the 2×2 to be written under the 3. By how many 10's must the 3 be multiplied to obtain a dividend such that we may cancel the two 2's of the divisor?

How many 0's, then, must be added to the 3?

Where must the decimal point be placed in the number thus produced that the value of the dividend of the fraction may not be changed?

3.00 divided by 4 equals what?

What decimal, then, is equivalent to the common fraction $\frac{3}{4}$?

We are to change to an equivalent decimal a common fraction whose denominator is 125.

What are the prime factors of 125?

How many 0's then, must be added to the dividend of the fraction to render it divisible by the divisor?

How many 0's would need be added if the divisor of a fraction were

5?	8?	625?	32?	128?
25?	2?	16?	64?	256?

We are to reduce to an equivalent decimal a common fraction whose denominator is 50.

What are the prime factors of 50?
Which of these factors will be cancelled by a single 10 in the dividend?
How many 0's, then, must be added to the dividend of the fraction?

How many 0's must be added to the dividend of a fraction whose divisor is

20?	80?	320?	250?
200?	400?	800?	2500?

How, then, does the number of 0's that must be added to the dividend of a common fraction compare with the number of 2's and 5's in its divisor?

We are to change $\frac{7}{15}$ to an equivalent decimal.
What are the prime factors of 15?
Will the multiplication of the dividend by 10, or by 10 taken any number of times as a factor, enable us to cancel the 3 from the divisor?
Can, then, $\frac{7}{15}$ be changed to an equivalent decimal?

What are the only factors of a divisor that can be cancelled by multiplying the dividend by 10?
What, then, are the only common fractions that can be changed to equivalent decimals?

SOLUTIONS.

Ex. 1. Reduce $\frac{3}{4}$ to a decimal.
Ex. 2. Reduce $\frac{7}{8}$ to a decimal.
Ex. 3. Reduce $\frac{13}{40}$ to a decimal.
Ex. 4. Reduce $\frac{7}{11}$ to a decimal.
Ex. 5. Reduce $\frac{7}{15}$ to a decimal.

(1)	(2)	(3)
.75	.875	.325

(4)	(5)
.636 +	$.466\frac{2}{3}$

EXPLANATIONS.

Ex. 1. We are to change $\frac{3}{4}$ to an equivalent decimal.

$4 = 2 \times 2$. Therefore, a dividend that will be divisible by 4 may be obtained by multiplying 3 by 10×10, or by reducing the 3 integers to hundredths. 3 integers equal 300 hundredths, or 3.00; and $3.00 \div 4 = .75$. Therefore, $\frac{3}{4} = .75$.

Ex. 2. Explain the solution of Ex. 2.

Ex. 3. We are to change $\frac{13}{40}$ to an equivalent decimal.

$40 = (5 \times 2) \times 2 \times 2$. Therefore, a dividend that will be divisible by 40 may be obtained by multiplying 13 by $10 \times 10 \times 10$, or by reducing the 13 integers to thousandths. 13 integers equal 13000 thousandths, or, 13.000; and $13.000 \div 40 = .325$. Therefore, $\frac{13}{40} = .325$.

Ex. 4. We are to reduce $\frac{7}{11}$ to a decimal.

The denominator of the fraction $\frac{7}{11}$ contains a factor other than 2 or 5; therefore, the fraction cannot be reduced to an exact decimal. We can, however, by adding 0's to the numerator carry out the indicated division to any desired number of decimal places. Extending our quotient to three decimal places, we find that $\frac{7}{11} = .636+$, or $.636\frac{4}{11}$.

Ex. 5. Explain the solution of Ex. 5.

NOTE 1. A fraction with the denominator 25, 125, etc., may most readily be reduced to a decimal by multiplying both numerator and denominator by such a number as will change the denominator to the required power of 10. Thus, $\frac{7}{25} = \frac{28}{100} = .28$; and $\frac{49}{125} = \frac{392}{1000} = .392$.

NOTE. 2. The decimal equivalents of fractions like $\frac{7}{11}$ and $\frac{7}{15}$ are called INFINITE or CIRCULATING decimals. Observe that in circulating decimlas the same figure or set of figures is constantly repeated. Thus, $\frac{7}{11} = .63\,63\,63$, to an infinite number of figures; and $\frac{7}{15} = .4\,666$, to an infinite number of figures.

The figure or set of fignres that is constantly repeated is called a REPETEND. It is evident that the end of a repetend has been reached when a remainder is found that is the same as a preceding remainder.

A decimal like $.466\frac{2}{3}$ may be referred to as a **Mixed Decimal.** If the decimal equivalent of a common fraction like $\frac{7}{15}$ is only approximately expressed it may be written. 467 instead of .466 +. Why?

Ex. 74.

1. State which of the fractions in the following exercise can be reduced to exact decimals. State also the number of 0's that must be added to the numerator of each fraction that can be so reduced, and verify your statement by performing the reduction.

2. Change the remaining fractions to approximately equivalent decimals. Continue the quotient either until a complete repetend has been found or to six decimal places.

1. $\frac{3}{4}$	6. $\frac{2}{7}$	11. $\frac{19}{24}$	16. $\frac{75}{240}$
2. $\frac{1}{2}$	7. $\frac{3}{5}$	12. $\frac{21}{24}$	17. $\frac{274}{625}$
3. $\frac{1}{3}$	8. $\frac{5}{6}$	13. $\frac{17}{25}$	18. $\frac{379}{640}$
4. $\frac{3}{8}$	9. $\frac{3}{6}$	14. $\frac{17}{32}$	19. $\frac{124}{240}$
5. $\frac{7}{8}$	10. $\frac{4}{9}$	15. $\frac{59}{60}$	20. $\frac{189}{360}$

59. To Change a Decimal to a Common Fraction.

We are to express .9357896 as a common fraction.

How many places would the period in the decimal need be moved to the right to change the decimal to the corresponding integer?

Removing the decimal point seven places to the right is equivalent to multiplying by 1 with how many 0's annexed?

Suppose, then, that we wish to change the given decimal to the corresponding integer. Our multiplier would consist of 1 with how many 0's annexed?

Suppose that we wish to change the corresponding integer to the given decimal. Our divisor would consist of 1 with how many 0's annexed.

What common fraction, then, is equivalent to the decimal .9357896?

By what would the corresponding integer need be divided to produce

.834?	.9476?	.05?	.304738?
.016?	.0083?	.95?	.000201?

How would the number of 0's in each divisor compare with the number of decimal places in the corresponding decimal?

The divisor, then, in each equivalent common fraction would be 1 with how many 0's annexed?

The dividend in each fraction would be what?

Give, then, a rule for changing a decimal to an equivalent common fraction.

What should be done in case the equivalent common fraction is not in its lowest terms?

Complete the rule for changing a decimal to an equivalent common fraction?

Ex. 1. Change .73458 to an equivalent common fraction.

Ex. 2. Change .0225 to an equivalent common fraction.

SOLUTIONS.

(1) $\frac{73458}{100000} = \frac{36729}{50000}$

(2) $\frac{225}{10000} = \frac{9}{400}$

EXPLANATIONS.

Ex. 1. We are to change .73458 to an equivalent decimal.

To change .73458 to an integer we should remove the decimal point 5 places to the right, or multiply by a multiplier containing 10 taken 5 times as a factor, or by 1 with 5 0's annexed. The given decimal, therefore equals the corresponding integer divided by 1 with 5 0's annexed, or 73458 ÷ 100,000, or $\frac{73458}{100000}$.

$\frac{73458}{100000}$ reduced to its lowest terms equals $\frac{36729}{50000}$. The answer to the exercise, therefore, is $\frac{36729}{50000}$.

Ex. 2. Explain the solution of Ex. 2.

Ex. 75.

Change to a common fraction in its simplest form

1. .5	11. .35	21. .375
2. .25	12. .48	22. .0475
3. .75	13. .64	23. .0024
4. $.16\frac{2}{3}$	14. .84	24. .525
5. .125	15. .45	25. .34864
6. $.12\frac{1}{2}$	16. .32	26. .240
7. $.333\frac{1}{3}$	17. .98	27. .144
8. $.6\frac{1}{4}$	18. .02	28. .156
9. $.87\frac{1}{2}$	19. .04	29. .8775
10. $.62\frac{1}{2}$	20. .12	30. .000018

60. Multiplication of Decimals.

Exercises in addition and subtraction of decimals, as we have already learned, depend for their solution upon the same principles as the corresponding exercises in addition and subtraction of integers.

As any decimal can be expressed as a common fraction, we would expect the principles governing multiplication and division of common fractions to also prevail in multiplication and division of decimals. That such is the case will be seen from a study of the inductive exercises in this and the following Articles.

* * *

We are to multiply .25 by .5.
Instead, we multiply 25 units by 5 units, thus obtaining as our product 125 units.
How does the multiplicand used compare with the correct multiplicand?
How, then, does the product, as affected by the error in the multiplicand, compare with the correct product?
How could we change the product so as to correct this error?
How do we divide a number by 100?

Explain in the same way the effect upon the product of the error in the multiplier.
How could the product, as affected by this error in the multiplier, be corrected?
How do we divide a number by 10?

How does the number of places that we would move the decimal point to the left to correct the error arising from the error in the multiplicand compare with the number of decimal places in the multiplicand?

How does the number of places that we would move the decimal point to the left to correct the error arising from the error in the multiplier compare with the number of decimal orders in the multiplier?

How, then, does the total number of places that we remove the decimal point to the left compare with the number of decimal orders in both multiplier and multiplicand?

Give, then, a rule for pointing off the product when the multiplier or multiplicand, or both, are decimals.

Give a general rule for multiplication of decimals.

	SOLUTIONS.		
	(1)	(2)	(3)
Ex. 1. Multiply 6.75 by .7.	4.725	75.76	945
Ex. 2. Multiply 947 by .08.			.08505
Ex. 3. Multiply .315 by .27.			

EXPLANATIONS.

Ex. 1. We are to multiply 6.75 by .7.

We first multiply 675 integers by 7 integers. The product thus obtained is 4725 integers.

The true multiplicand is not 675 integers, but 675 integers divided by 1 with two 0's annexed; and the true multiplier is not 7 integers, but 7 integers divided by 1 with one 0 annexed. Our true product, therefore, is the product already obtained divided by 1 with two plus one, or three, 0's annexed.

To divide 4725 by 1 with three 0's annexed, we remove the decimal point three places to the left. The quotient thus obtained, and the answer to the exercise, is 4.725.

Explain the solutions of Exs. 2 and 3.

Ex. 76.

Find the product of

1. $342.578 \times .34$
2. 842.38×73.4
3. 3.45937×24.3
4. 784.50×342.5
5. $3456.453 \times .234$
6. $9.4568 \times .02$
7. 74.578×34.56
8. $525.6 \times .156$
9. 8427.8×3.46
10. 98.975×37.6
11. 572.563×46.9
12. 74.5116×4.203
13. 375.810×6.8
14. 78.592×6.8
15. 732.554×4.527
16. $634.262 \times .243$
17. 746.264×3.278
18. 669.345×27.58
19. 59.7274×73.245
20. 2079.73×8.008
21. 456.357×50.3
22. 573.034×8.567
23. 42.21×1.907
24. 90.81×2.108

61. Division of Decimals.

We are to divide .864 by .08.

What is the effect of moving a decimal point a given number of places to the right?

How many places must we move to the right the decimal point in our divisor to change the .08 to 8 integers?

What change must we make in the position of the decimal point in our dividend to counterbalance the change in our divisor?

What is the final dividend thus produced?

What was our final divisor?

86.4 divided by 8 equals what?

What, then, is the answer to the exercise?

We are to divide .016 by .000004.

How many places must we remove our decimal point to the right to change our divisor to an integer?

What change, then, must be made in the position of the decimal point in the dividend?

How many figures are there at the right of the decimal point in the dividend?

What figures, then, must be added to the dividend to make possible the removal of the decimal point the required number of places?

16^{000} divided by 4 equals what?

What, then, is the answer to the exercise?

Add to the rule previously given so as to provide for division when there are more decimal places in the divisor than in the dividend.

SOLUTIONS.

Ex. 1.	Divide 1.6^{00} by .008·.	(1)	(2)	(4)
Ex. 2.	Divide .2·9 by .3·.	200	.967	
Ex. 3.	Divide 43.27 by .125.		(3)	
Ex. 4.	Divide 3771.36^{0} by .432·.		346.16	

$$\begin{array}{r} 8730 \\ 3771360 \\ \underline{3456} \\ 3153 \\ \underline{3024} \\ 1296 \\ 1296 \end{array}$$

EXPLANATIONS.

Ex. 1. We are to divide 1.6 by .008.

We first change our divisor to an integer by removing the decimal point three places to the right. To counterbalance this change in the divisor we remove three places to the right the decimal point in our dividend. The new dividend thus produced is 1600, and the divisor is 8.

1600 divided by 8 equals 200. Therefore, the quotient of 1.6 divided by .008, and the answer to the exercise, is 200.

Ex. 2. We change our divisor to an integer and make the corresponding change in our divisor, as in Ex. 1.

Dividing our new dividend, 2.9, by our new divisor, 3, and continuing the division until all the figures of the dividend have been used, we obtain the quotient .9 and the remainder .2.

2 and 3 contain no common factor, and no number of 0's annexed to the 2 will produce a dividend divisible by 3. We however obtain an approximate quotient by mentally annexing 0's to the dividend, and continuing the division through three decimal orders. The quotient thus obtained, and the answer to the exercise, is .967.

Ex. 3. We are to divide 43.27 by .125.

.125 equals $\frac{1}{8}$. 43.27 divided by $\frac{1}{8}$ equals 346.16. Therefore, the quotient of 43.27 divided by .125, and the answer to the exercise, is 346.16.

Ex. 4. Explain the solution of Ex. 4.

NOTE. 1. A rule for placing the point in the quotient without changing the divisor and dividend is developed in the following inductive exercises. Point off your quotient according to this rule if you prefer it to the one you have previously given.

What relation has a dividend to the corresponding divisor and quotient?

How does the number of decimal places in a product compare with the number in its factors?

How, then, does the number of decimal places in a dividend compare with the number of places in the corresponding divisor and quotient?

How, then, does the number of decimal places in a quotient compare with the number of places in the corresponding divisor and dividend?

Give, then, a rule for division of decimals.

In applying this rule increase the number of decimal places in the dividend if it has fewer places than the divisor.

NOTE. 2. In the following exercises, in determining how far each division shall be carried observe the following directions:

1. If by annexing 0's a dividend can be formed that will exactly contain the divisor, continue the division until there is no remainder.

2. If the dividend and divisor are so related to each other that the division will continue indefinitely,

(1) Continue the division at least until all the figures of the dividend have been used.

(2) Extend the quotient to at least two decimal places.

Ex. 77.

Divide

1. 3.75 by .5
2. 6.94 by .006
3. 94.2 by .6
4. 80.7 by .003
5. 7.80 by 1.2
6. 7.45 by .5
7. 64.8 by .008
8. 7.48 by .7
9. 91.9 by .04
10. 90.34 by .6
11. 7625 by .5
12. 9.076 by .08
13. 46267 by .7
14. 6.852 by .06
15. .6075 by .4
16. 283.4 by .07
17. 2.176 by .09
18. 3247 by .3
19. 307.4 by .09
20. .9439 by .007
21. 805.7 by .4
22. 9843 by .003
23. .2176 by .4
24. 532.8 by .06
25. 6.674 by .008
26. 576.5 by .012
27. 9.823 by .3

Ex. 78.

Divide

1. 3674.56 by 745.68
2. 32598 by 33.42
3. 78.923 by 621.4
4. 69247 by 33.21
5. 7893.1 by 63.145
6. 312.46 by 342
7. 1745.64 by 221.4
8. 35.9784 by 90.7
9. 3745.6 by 93.8
10. 34.593 by 75.369
11. 7897.3 by 35.41
12. 372.5 by 70.375
13. 634.59 by 334.5
14. 7842.7 by 3.3462
15. 46357 by 33.21
16. 345.22 by 11.19
17. 1714.16 by 3.22
18. 56.3456 by 73.805
19. 424.127 by 231.5
20. 94.3221 by 1.643
21. 988.34 by 502.6
22. 524.795 by 3.61
23. 9583.7 by 3.324
24. 305.75 by 42.3

Ex. 79.

Divide

1. 3459.7636 by 7.503
2. 26485.347 by 3.842
3. 426873.34 by .2451
4. 9876.6214 by 1.142
5. 9.4213754 by 226.3
6. 254637.45 by 7.546
7. 45.376421 by 34.16
8. 294567.43 by 274.5
9. 243.45634 by .3345
10. 3459.8734 by 756.3
11. 987.35142 by 247.0
12. 40525.608 by 5.865
13. 508.00502 by 26.04
14. 852374.34 by 150.4
15. 45.637250 by 202.5
16. 6521.2774 by 58.37
17. 5.6327813 by 2.783
18. 683057.52 by .9873
19. 59.622784 by .3209
20. 256.37541 by 21.75
21. 26356.714 by 22.35
22. 463.27883 by 16.54
23. .51742143 by 247.2
24. 356.37517 by 35.96
25. 2956889.4 by .0064
26. 3.0546189 by 500.4

Extract from Monthly Statement of Claremont Creamery.

	George Stevens	Frank Hamlin	James Brown	Henry Lane	Alfred Davis	Edward Miller
	Pounds Milk	Pounds Milk	Pounds Milk	Pounds Milk	Pounds Cream	Pounds Cream
Feb. 1	129	146	229	22	107¾	56½
" 2	137	77	138	14		
" 3	121	77	133	16		
" 4	249	85	120	16		
" 5	133	85	131	27	62	109¼
" 6	139	80	134	15		
" 7	148	74	133	12		
" 8	151	72	133	15		
" 9	144	74	119	14	43¾	56½
" 10	155	71	147	13		
" 11	142	71	152	13		
" 12	149	66	169	15		
" 13	141	67	157	15	68¼	81¾
" 14	156	59	110	14		
" 15	143	71	180	17		
" 16	142	73	82	16		
" 17	137	71	145	17	55¼	61
" 18	147	74	199	15		
" 19	138	76	137	17		
" 20	157	75	136	17		
" 21	140	73	158	17	66¾	89
" 22	131	75	125	13		
" 23	141	75	145	14		
" 24	131	77	120	13	62¼	62¼
" 25	130	79	135	18		
" 26	111	79	192	17		
" 27	136	84	85	18		
" 28	130	83	180	16	56½	82¾
" 29	148	76	137	17		

(Creamery Envelope)

Claremont Creamery Association.

Month of ____ ______

Name ____________

Number Pounds Milk Furnished .... ________

Average Per Cent Test .... ________

Number Pounds Butter Fat ____ at ____ ... $________

(Weekly Test)

	First week	Second week	Third week	Fourth week	Average
Stevens (Milk)	4.45	4.05	4.2	4	( )
Hamlin (Milk)	4.9	5.1	5.05	4.9.	( )
J. Brown (Milk)	4.95	3.90	3.85	3.7	( )
Lane (Milk)	5.7	5.6	5.65	5.01	( )
Davis (Cream)	22.5	24	21.5	25.5	( )
Miller (Cream)	15.5	15	14.75	15.5	( )

Problems.

1. Find the average monthly test of the milk or cream of each of the six patrons?

2. Draw up six blank statements like that given on the front of the creamery envelope, and fill out one for each of the six patrons, the selling price of the butter fat being 23 c.

3. By what per cent would the monthly income of Brown have been increased had the average monthly test of his milk been equal to that of Lane's.

4. By what per cent would Lane's income have been diminished had the test of his milk been only equal to that of Stevens'?

5. What would have been the per cent of increase in income had the average test of the milk of the first four patrons been equal to the highest weekly test of any one.

NOTE. Form and solve problems like the last three till all the principles involved have been perfectly mastered. More involved problems pertaining to creameries will be given under corporations.

62. Products from Two times Thirteen to Nine times Nineteen.

The following table includes all products of which one factor is between 13 and 19 and the other between 2 and 9. Practise upon the table until you can mentally complete it with rapidity and correctness.

is 8 times 16	78 is times 13	144 is times 16
112 is 7 times	117 is 9 times	64 is 4 times
is 7 times 13	96 is 6 times	162 is times 18
136 is times 17	26 is times 13	45 is 3 times
is 3 times 13	is 9 times 17	85 is 5 times
76 is times 19	38 is 2 times	26 is 2 times
is 5 times 15	52 is times 13	117 is times 13
is 9 times 19	is 3 times 14	is 6 times 19
60 is 5 times	is 8 times 19	72 is times 18
114 is 6 times	32 is 2 times	128 is times 16
112 is 8 times	is 4 times 19	is 8 times 17
is 9 times 14	144 is 8 times	38 is times 19
is 5 times 14	is 8 times 18	is 4 times 15
90 is times 15	39 is times 13	102 is 6 times
51 is 3 times	171 is times 19	112 is 8 times
is 9 times 15	60 is 4 times	90 is times 18
is 5 times 18	is 2 times 17	57 is 3 times
126 is times 14	126 is times 18	68 is times 17
is 5 times 13	105 is times 15	126 is 9 times
is 2 times 19	is 3 times 15	is 6 times 13
152 is 8 times	144 is times 18	108 is times 18
is 3 times 17	68 is 4 times	133 is 7 times
90 is 6 times	152 is times 19	is 8 times 13
is 4 times 13	is 8 times 14	is 3 times 19
is 9 times 18	is 6 times 14	48 is times 16
120 is times 15	95 is times 19	95 is times
72 is 4 times	is 8 times 14	is 2 times 14
is 7 times 19	is 6 times 14	135 is times 15
76 is 4 times	95 is 5 times 19	114 is times 19
135 is 9 times	is 7 times 17	28 is 2 times
70 is 5 times	34 is 2 times	119 is times 17
108 is 6 times	56 is 4 times	78 is 6 times
is 2 times 13	126 is 7 times	is 7 times 16
144 is 9 times	is 4 times 14	30 is 2 times

65 is times 13	is 5 times 19	171 is 9 times
52 is 4 times	70 is times 14	104 is 8 times
30 is times 15	is 3 times 19	91 is 7 times
133 is times 19	162 is 9 times	105 is 7 times
is 9 times 17	90 is 5 times	54 is times 18
75 is 5 times	84 is 6 times	42 is 3 times
34 is times 17	102 is times 17	36 is 2 times
39 is 3 times	is 5 times 17	56 is times 14
is 9 times 16	75 is times 15	is 2 times 16
80 is times 16	is 4 times 17	80 is times 16
is 7 times 18	36 is times 18	98 is 7 times
98 is times 14	is 3 times 18	80 is 5 times
is 6 times 18	is 4 times 17	98 is 7 times
64 is times 16	45 is times 15	is 6 times 16
45 is times 15	is 6 times 16	is 6 times 15
112 is times 16	is 6 times 15	is 7 times 14
84 is times 14	is 3 times 16	32 is times 16
136 is 8 times	is 8 times 15	153 is times 17
48 is 3 times	54 is 3 times	120 is 8 times
85 is times 17	is 7 times 15	is 5 times 16
is 2 times 15	is 4 times 16	128 is 8 times

NOTE. In multiplying, for example, 17 by 7, (1) multiply the 7 by 7 and mentally record the product; (2) multiply the 1 by 7, and add the left-hand figure of the preceding product; and (3) give orally, or write, the total product. Follow the same plan in the exercises in division. As will be seen from the preceding paragraph, it is not necessary to memorize the different products and quotients. These will, however, without conscious effort, be gradually memorized if the pupil always performs the exercises by mental processes.

Ex. 80.

Multiply	Multiply	Divide
1. 4213 by 19	10. 8736 by 17	19. 84546 by 15
2. 8736 by 13	11. 4583 by 17	20. 97342 by 19
3. 4923 by 15	12. 9327 by 16	21. 24386 by 13
4. 6897 by 14	13. 8543 by 19	22. 29460 by 17
5. 2543 by 18	14. 9248 by 14	23. 47824 by 16
6. 7346 by 16	15. 2435 by 18	24. 39476 by 18
7. 9834 by 13	16. 5482 by 19	25. 29657 by 19
8. 3495 by 17	17. 2678 by 15	26. 43265 by 14
9. 2759 by 14	18. 9687 by 13	27. 24832 by 16

63. To Multiply without Writing All the Partial Products.

We are to multiply 763 by $3\frac{5}{7}$. 545
We first multiply by $\frac{5}{7}$. 2834
What product do we thus obtain?

We next multiply 3 by 3. What product do we thus obtain?
What figure of the first product must be combined with this product?
9 plus 5 equals what?
What, then, is the right-hand figure of the total product?
What must be done with 1, the left-hand figure of the first partial product by 3?

We next multiply the 6 by 3.
What product do we thus obtain?
What two figures must be combined with the 18?

What amount do we thus obtain?
What, then, is the second figure of the total product
What shall we do with 2, the left-hand figure of the amount?

We next multiply the 7 by 3. What product do we thus obtain?
What two figures must be combined with this product?
What amount do we thus obtain?
What, then, are the two left-hand figures of the total product?
What, then, is the total product?
Give, then, a special rule for multiplying an integer by a mixed number when the integral part of the mixed number is not greater than 19.

We are to multiply 4873 by 53 without writing down the second partial product. 14619
258269
We first multiply by 3, and next write down the 9 as the first right-hand figure of the total product.
How do we find the second figure of the total product?
Why do we combine the right-hand figure of our product by the 5 with the 1 of the first partial product instead of with the 9?

How do we find the third figure of the total product? the fourth? the fifth and sixth figures?
Give a special rule for multiplying by a multiplier of two figures.

	SOLUTIONS.		
	(1)	(2)	(3)
Ex. 1. Multiply $\overset{1670}{334}$ by $7\frac{5}{8}$.	$208\frac{3}{4}$	5733	39767
Ex. 2. Multiply 819 by 57.	$2546\frac{3}{4}$	46683	
Ex. 3. Multiply 437 by 91.			

EXPLANATIONS.

Ex. 1. We are to multiply 334 by $7\frac{5}{8}$.

We first multiply 334 by $\frac{5}{8}$. The product thus obtained is $208\frac{3}{4}$.

We next multiply 334 by 7 Instead, however, of writing down the partial products, we combine each as we obtain it with the proper figure of the first partial product. We thus obtain as our final product $2546\frac{3}{4}$

Ex. 2. Explain the solution of Ex. 2.

Ex. 3. We are to multiply 437 by 91.

Once 437 is 437. Our first partial product, therefore, is 437.

We next multiply 437 by 9, and, as we multiply, combine each partial product with the proper figure of the first product. We thus obtain as our final product 39767.

Ex. 81.

Find the product of

1. $7826 \times 9\frac{1}{2}$	11. 2365×23	21. 582×81
2. $9485 \times 7\frac{3}{8}$	12. 8761×47	22. 979×23
3. $4934 \times 8\frac{4}{9}$	13. 6376×53	23. 386×53
4. $2547 \times 9\frac{4}{7}$	14. 8174×41	24. 874×76
5. $8436 \times 6\frac{7}{13}$	15. 9376×73	25. 948×38
6. $4378 \times 9\frac{11}{15}$	16. 8432×67	26. 824×61
7. $3236 \times 7\frac{17}{19}$	17. 2479×87	27. 467×47
8. $5374 \times 8\frac{13}{15}$	18. 5234×91	28. 286×38
6. $7347 \times 5\frac{12}{17}$	19. 4376×83	29. 752×43
10. $8216 \times 6\frac{13}{18}$	20. 7247×93	30. 853×57

64. To Prove a Multiplication by Eliminating 9's or 11's.

We have multiplied 8744 by 638, and wish to test the correctness of our work by eliminating 9's.

( ) nines + 5
( ) nines + 8

Under Divisibility of Numbers, we learned that every number is composed of a certain number of nines plus the sum of the figures of the number. Our multiplicand, then, is composed of a certain number of 9's plus what? Our multiplier?

23, the sum of the figures of the multiplicand, is composed of a certain number of nines plus what?

Our multiplicand, 8749, may, then, be thought of as a certain number of 9's plus what? Our multiplier, 638?

In multiplying a certain number of 9's plus 5, by a certain number of nines plus 8, what is the first partial product? the second? the third? the fourth?

Which of these products are composed entirely of a certain number of 9's?

The remaining partial product is the product of what factors?

5 times 8 equals a certain number of 9's plus what?

The first partial product, then, equals a certain number of 9's plus what?

The total product, then, equals a certain number of 9's plus what?

Two factors, then, being given, the number other than a certain number of 9's that will be a part of the product may be how obtained before the multiplication itself is performed?

How may this number be obtained after the multiplication is performed?

Suppose that these two numbers do not agree. What follows as to the correctness of the operations that have been performed?

Give, then, a rule for proving a multiplication through the elimination of 9's.

Suppose that an error is made of 9 or any number of 9's. Will such an error be shown by this test?

Add to your rule a statement showing the limit of its application.

What relation has a dividend to its divisor and quotient?

Modify, then, the preceding rule so that it may be used in proving a division.

What number other than a certain number of 9's is a part of the product when the factors are

3622 and 8395?	3055 and 2428?	36026 and 13504?
6027 and 4209?	5720 and 2417?	92066 and 52809?
8982 and 2496?	4062 and 6329?	41095 and 89462?
9427 and 4529?	4292 and 7180?	31906 and 79126?

* * *

We have multiplied 9157 by 436, and wish to test the correctness of our work by eliminating 11's.

( ) elevens + 5
( ) elevens + 7

We have learned that any number is composed of a certain number of 11's plus the sum of its odd figures, commencing at the right, and minus the sum of its even figures. 9157, then, may be thought of as a certain number of 11's minus what number?

If 9157 is a certain number of 11's minus 6, it is a one less number of 11's plus what number?

What expression, then, may we use to represent our multiplicand?

What expression do we use to represent our multiplier?

Explain in full the multiplication of the two expressions.

Give a rule for proving a multiplication by eliminating 11's.

Give a rule for proving a division by eliminating 11's.

What number other than a certain number of 11's will be a part of the product of each pair of factors previously given on this page?

61. Division of Decimals.

We are to divide .864 by .08.

What is the effect of moving a decimal point a given number of places to the right?

How many places must we move to the right the decimal point in our divisor to change the .08 to 8 integers?

What change must we make in the position of the decimal point in our dividend to counterbalance the change in our divisor?

What is the final dividend thus produced?

What was our final divisor?

86.4 divided by 8 equals what?

What, then, is the answer to the exercise?

We are to divide .016 by .000004.

How many places must we remove our decimal point to the right to change our divisor to an integer?

What change, then, must be made in the position of the decimal point in the dividend?

How many figures are there at the right of the decimal point in the dividend?

What figures, then, must be added to the dividend to make possible the removal of the decimal point the required number of places?

16^{000} divided by 4 equals what?

What, then, is the answer to the exercise?

Add to the rule previously given so as to provide for division when there are more decimal places in the divisor than in the dividend.

Ex. 1. Divide 1.6^{00} by $.008^{\cdot}$.

Ex. 2. Divide $.2{\cdot}9$ by $.3^{\cdot}$.

Ex. 3. Divide 43.27 by .125.

Ex. 4. Divide 3771.36^{0} by $.432^{\cdot}$.

SOLUTIONS.

(1) 200

(2) .967

(3) 346.16

(4)

$$\begin{array}{r} 8730 \\ 3771360 \\ \underline{3456} \\ 3153 \\ \underline{3024} \\ 1296 \\ 1296 \end{array}$$

EXPLANATIONS.

Ex. 1. We are to divide 1.6 by .008.

We first change our divisor to an integer by removing the decimal point three places to the right. To counterbalance this change in the divisor we remove three places to the right the decimal point in our dividend. The new dividend thus produced is 1600, and the divisor is 8.

1600 divided by 8 equals 200. Therefore, the quotient of 1.6 divided by .008, and the answer to the exercise, is 200.

Ex. 2. We change our divisor to an integer and make the corresponding change in our divisor, as in Ex. 1.

Dividing our new dividend, 2.9, by our new divisor, 3, and continuing the division until all the figures of the dividend have been used, we obtain the quotient .9 and the remainder .2.

2 and 3 contain no common factor, and no number of 0's annexed to the 2 will produce a dividend divisible by 3. We however obtain an approximate quotient by mentally annexing 0's to the dividend, and continuing the division through three decimal orders. The quotient thus obtained, and the answer to the exercise, is .967.

Ex. 3. We are to divide 43.27 by .125.

.125 equals $\frac{1}{8}$. 43.27 divided by $\frac{1}{8}$ equals 346.16. Therefore, the quotient of 43.27 divided by .125, and the answer to the exercise, is 346.16.

Ex. 4. Explain the solution of Ex. 4.

NOTE. 1. A rule for placing the point in the quotient without changing the divisor and dividend is developed in the following inductive exercises. Point off your quotient according to this rule if you prefer it to the one you have previously given.

What relation has a dividend to the corresponding divisor and quotient?

How does the number of decimal places in a product compare with the number in its factors?

How, then, does the number of decimal places in a dividend compare with the number of places in the corresponding divisor and quotient?

How, then, does the number of decimal places in a quotient compare with the number of places in the corresponding divisor and dividend?

Give, then, a rule for division of decimals.

In applying this rule increase the number of decimal places in the dividend if it has fewer places than the divisor.

NOTE. 2. In the following exercises, in determining how far each division shall be carried observe the following directions:

1. If by annexing o's a dividend can be formed that will exactly contain the divisor, continue the division until there is no remainder.

2. If the dividend and divisor are so related to each other that the division will continue indefinitely,

(1) Continue the division at least until all the figures of the dividend have been used.

(2) Extend the quotient to at least two decimal places.

Is .4837 nearer .483 or .484 ?

What figure, then, shall we use as the right-hand figure of our first multiplicand?

How does the order of the second figure of the multiplier compare with the order of the first?

How, then, to obtain the same order in the product as in the first multiplication, must the order of the right-hand figure of the mulitiplicand in the second multiplication compare with the order of the corresponding figure in the first?

What figure, then, shall we use as the right-hand figure of the multiplicand in the second multiplication?

Explain in full the determining of the right-hand figure of the multiplicand in the third multiplication.

Are there any figures in the multiplicand of a sufficiently high order to multiply by the remaining figure of the multiplier?

What is the remaining step of the operation?

Give, then, a rule for obtaining a product in multiplication of decimals approximately correct to any decimal order.

* * *

We are to divide .17013453508 by 680.97, and to make use of a sufficient number of figures in the dividend and divisor so that the product of the divisor and the quotient will approximately equal the dividend to millionths.

What will be the first step in the operation?

What will be the position of the decimal point in the new dividend?

What are the fewest figures of the dividend that will contain the divisor ?

What, then, will be the order of the first quotient figure?

A fourth order × a second order gives what order?

How will the order of the second quotient figure compare with the order of the first?

What figure, then, must we use as the last figure of the divisor in the second division that its product by the quotient may be of the desired order?

Explain in full what figure will be used as the last figure of the divisor in the third division? in the fourth division? in the fifth division?

SOLUTIONS.

Ex. 1. Find the approximate product to hundredths of 314.65 multiplied by .345.

Ex. 2. Find the quotient of 34.125487 divided by 8.574. Let the partial products be at least approximately correct to ten-thousandths.

(1)

```
   94·41
 · 12 60
    1 55
  108 56
```

(2)

```
        3.9801
  34125·487
  25722
   84034
  77166
    6869
    6856
      13
       9
```

EXPLANATIONS.

Ex. 1. We first multiply by the left-hand figure of the multiplier. This figure is of the first decimal order. The product is to be of the second decimal order. The right-hand figure of the first multiplicand, therefore, must be of the first order, and we begin our multiplication by multiplying the tens of the multiplicand by the left-hand figure of the multiplier.

We next multiply in order by the figures at the right of the left-hand figure. As each figure of the multiplier is one order lower than the preceding, the right-hand figure of each new multiplicand must be one order higher than the preceding.

We continue our multiplication until all the figures of the multiplier have been used. We next add our partial products, and obtain as our final product 108.56

Ex. 2. We first move the decimal point in our dividend aud divisor places enough to the right to change the divisor to an integer. Our new divisor is 8574, and our new dividend 34125.487,

We divide in the ordinary manner until our quotient has been extended to two decimal orders. As there are three decimal orders in our divisor, and as we do not wish to use partial dividends of more than four decimal orders, we annex no more figures to our partial dividend, but, instead, reject at each division the right-hand figure of the divisor used in the preceding division. Continuing our division until no figure of our divisor remains, we have as our quotient 3.9801.

Ex. 83.

Obtain the approximate product to thousandths in the first and each third exercise in Ex. 76. Solve the first and each fifth exercise in Ex. 79. Let no partial dividend be of a lower order than thousandths.

66. To Convert Factors into More Convenient Factors.

We wish to find the product of the factors 137, 32, and 2, and to do this with only two multiplications.

Suppose that we subtitute the factor 8 for the factor 2. The new factor will be how many times as large as the original factor?

What change can we make in the factor 32 that will counterbalance the change in the smaller factor?

What two more convenient factors, then, may take the the place of 32 and 2?

Give, then, a rule for subtituting more convenient factors for two or more given factors.

Ex. 1. Find the product of $733 \times 35 \times 4 \times 2 \times 24 \times 125$.

SOLUTION.

51310
615720
4925760
615720000

EXPLANATION.

Twice 35 are 70, and 70 times 733 are 51310.

For the factors 4 and 24, we substitute 8 and 12. 12 times 51310 are 615720, and 8 times 615720 are 4925760.

We next multiply by 125 by mentally annexing three 0's and dividing by 8. The final product thus obtained is 615720000.

Ex. 84.

Find the product of

1. $75 \times 36 \times 2 \times 18 \times 5 \times 25 \times 5$
2. $834 \times 11 \times 3 \times 3 \times 35 \times 6 \times 50$
3. $93 \times 75 \times 8 \times 48 \times 3 \times 15 \times 8$
4. $86 \times 40 \times 24 \times 6 \times 16 \times 5 \times 18$
5. $93 \times 20 \times 6 \times 15 \times 18 \times 6 \times 15$

67. To Multiply 63 by 67, 54 by 56, etc.

We are to multiply 63 by 67.

What do we observe as to the relation to each other of the left-hand figures of the two factors?

What as to the sum of the right-hand figures?

Suppose that we perform the multiplication. What two figures will be the factors of our first partial product? of our second partial product? of our third partial product? of our fourth partial product?

7 units of 6's plus 3 units of 6's are how many tens of 6's?
6 tens of 6's plus 1 ten of 6's are how many tens of 6's?

What is the order of the 6?

Tens multiplied by tens give what order?

How many places, then, must be occupied by the first partial product?

How, then, may we obtain the two right-hand figures of the total product?

What product will form the third and fourth orders?

What, then, is the total product?

Give, then, a special rule for finding the product of two factors composed of two orders when the left-hand figures are the same and the sum of the right-hand figures is 10?

Ex. 1. Multiply 247 by 243.

SOLUTION.

60021

EXPLANATION.

We are to multiply 247 by 243.

We observe that the tens in each factor are the same, and that the sum of the units is 10. We therefore can perform the multiplication without writing all the partial products.

We observe that the sum of the three last partial products will contain no significant figure of the second order. We therefore write 21 as the two right-hand figures of our total product.

The second and the third partial products consist of 7 units of 24 tens plus 3 units of 24 tens, or of 1 ten of 24 tens. The last partial product is 24 tens of 24 tens. The last three partial products, therefore, are 24 plus 1, or 25, tens of 24 tens, or 600 hundreds, and the total product is 600 hundreds plus 21, or 60021.

Without recording the partial products, write the products of

37 and 33	72 and 78	43 and 47	996 and 994
41 and 49	86 and 84	26 and 24	492 and 498
16 and 14	55 and 55	34 and 36	241 and 249
46 and 44	63 and 67	91 and 99	1243 and 1247

68. To Divide in Long Division without Writing the Partial Products.

We wish to divide 2175 by 87 without writing down our partial products.

```
  25
2175
 435
```

What is our first partial dividend?
What is our first quotient figure?
$7 \times 2 = ?$ $7 - 4 = ?$
What, then, are the right-hand figures of our second partial dividend?

$8 \times 2 = ?$ What have we to add to this product from the preceding product? $16 + 1 = ?$ $21 - 17 = ?$ What, then, is the remaining figure of our second partial dividend?

Explain the remaining steps of the operation.
Give a rule for dividing without bringing down the partial products.

SOLUTION.

Divide 60012654 by 183 without writing down the partial product.

```
   327938
 60012654
 511
  1452
   1716
    695
    1464
```

EXPLANATIONS.

We divide according to the method ordinarily followed, except that we do not write down the product of a figure of the divisor by a quotient figure, but subtract it as we obtain it. Following this method, and using special care in making the necessary corrections when a figure of the minuend has been increased by 10, we obtain as our quotient 327938.

Ex. 85.

Solve, without writing the partial products, the first and every fifth exercise in Ex. 44.

Table Showing Area and Population of Principal Countries of the World.

	Government	Area	Population
Asia . . .	Grand Division	17,039,066	825,954,000
Africa . . .	" "	11,518,104	168,499,017
North America .	" "	9,000,000	88,175,900
South America .	" "	7,756,900	33,344,490
Europe. . .	" "	3,797,410	357,851,580
Oceania . .	" "	3,458,029	5,684,600
Asiatic Russia .	Empire	6,564,778	22,697,469
China . . .	"	4,218,401	402,680,000
United States .	Republic	3,602,990	62,622,250
British America.	Geog. Division	3,556,350	5,083,316
Canada. . .	British Colony	3,456,383	4,833,239
Brazil . . .	Republic	3,209,878	14,333,915
Russia in Europe	Empire	2,095,616	106,191,795
Argentine Rep.	Republic	1,778,195	3,954,911
Turkey. . .	Empire	1,576,700	38,791,000
India . . .	"	1,559,603	287,123,350
South Australia.	British Colony	903,690	320,431
Mexico. . .	Republic	767,005	12,619,959
Persia . . .	Kingdom	628,000	7,653,000
Venezuela . .	Republic	593,943	2,323,527
Bolivia. . .	"	567,430	2,019,549
Colombia . .	Republic	504,773	3,878,600
Peru . . .	"	463,747	2,621,844
Egypt . . .	Turkish Trib. State	400,000	9,734,405
Tripoli. . .	Turkish Province	398,900	1,300,000
New South Wales	British Colony	310,700	1,132,230
Chile . . .	Republic	293,970	2,712,145
Austria-Hungary	Monarchy	240,942	41,358,886
Germany . .	Empire	208,830	52,279,915
France. . .	Republic	204,092	38,343,192
Spain . . .	Kingdom	197,670	17,565,632
Sweden . .	"	172,878	4,784,981
Central America	Geog. Division	172,700	3,231,400
Guiana. . .	Colonies	170,500	373,900
Japan . . .	Empire	161,245	44,212,429
Norway . .	Kingdom	124,445	2,917,000
British Isles .	"	120,979	38,104,975
Ecuador . .	Republic	120,000	1,271,861
Philipine Islands	American Colony	114,326	7,000,000
Italy . . .	Kingdom	110,646	31,667,946
New Zealand .	British Colony	104,471	626,658
Paraguay . .	"	98,000	432,000
West Indies .	Geog. Division	96,550	5,483,900
Victoria . .	British Colony	87,884	1,140,405
Uruguay . .	Republic	72,110	827,485

	Government	Area	Population
England . .	Kingdom	50,823	27,483,490
Nicaragua . .	Republic	49,200	380,000
Roumania . .	Kingdom	48,307	5,800,000
Greenland . .	Danish Colony	46,740	10,516
Honduras . .	Republic	43,000	400,000
Newfoundland .	British Colony	42,200	202,000
Cuba . . .	U. S. Protectorate	41,655	1,631,687
Iceland . .	Danish Colony	39,756	70,927
Bulgaria . .	Principality	38,080	3,310,713
Portugal . .	Kingdom	34,528	4,660,095
Ireland. . .	"	32,531	4,704,750
Scotland . .	"	30,902	4,025,647
Tasmania . .	British Colony	26,385	146,667
Greece . . .	Kingdom	25,014	2,433,806
Costa Rica . .	Republic	23,000	243,205
Servia . . .	Kingdom	19,050	2,312,484
Switzerland. .	Republic	15,976	2,917,754
Denmark . .	Kingdom	15,289	2,185,335
Liberia . . .	Republic	14,360	1,068,000
The Netherlands	Kingdom	12,648	5,004,204
Belgium . .	"	11,373	6,586,593
Haiti . . .	Republic	10,204	960,000
Wales . . .	Kingdom	7,363	1,519,035
Hawaii . . .	U. S. Territory	6,740	109,020
Jamaica . .	British Colony	4,424	648,558
Puerto Rico. .	U. S. Colony	3,670	806,708
Montenegro .	Principality	3,630	228,000
Luxemburg . .	Grand Duchy	998	217,583
Andorra . .	Republic	175	6,000
San Marino . .	"	32	8,500
Monaco . .	Principality	8	13,304
Gibraltar . .	Colony	1.9	26,203

SUGGESTIONS TO TEACHERS. The following is one of the many uses that in connection with the study of geography may be made of the preceding table.

Select a number of countries, ten, for example, part of which are remarkable for the density and part for the sparseness of their population. Let your pupils find the density of population of each, and associate each result with the age of the country, its surface, its climatic conditions, and other important and pertinent facts. As a result, geography will appeal to them with new force in its true sphere as the study of the world as the abode of man, and will be pursued with increased interest and zeal.

NOTE. In all operations in the following problems that involve decimals, complete all finite decimals, and extend all others to thousandths.

Problems in Integers and Decimals.

1. Find the area of the six Grand Divisions.

2. Find the population of the six Grand Divisions.

3. Find the density of population of the world, and the density of population of each of the six Grand Divisions.

4. What would be the population of North America if its density of population were that of Europe.

What would be the population of the United States if its density of population were that of Belgium?

6. What is the ratio of the area of France to the area of the United States? of the area of France to the area of Spain? of the area of France to the area of Germany? of the area of France to the area of Sweden? of the area of France to the area of Austria Hungary? of the area of France to the area of Greece? of the area of France to the area of England? of the area of France to the area of Russia?

7 What is the ratio of the population of France to that of each of the other countries named in Problem 6?

8. Make a table showing the density of population of each of the nine countries, and having the countries arranged according to the relative densities.

9. Find the ratio of the density of population of the United States to the density of population of the Philippine Islands.

What would be the population of the Philippine Islands if their density of population were that of the United States?

What would be the population of the United States if its density of population were that of the Philippine Islands?

[In solving the first part of the preceding problem, express the density of population of each country in the form of a fraction, divide the first fraction by the second, and express the quotient as a decimal.

In solving any problem in which the only operations involved are multiplication and division, it is ordinarily best to indicate all operations before performing any of them.]

10. Find the ratio of the density of population of England to that of the United States; to that of South Australia; to that of Venezuela; to that of Paraguay: to that of Ireland; to that of India; to that of Canada; to that of Switzerland; to that of Japan.

11. What would be the population of Norway if its densisity of population were that of England?

12. What is the ratio of the density of population of the United States to that of Mexico? to that of China? to that of Tripoli? to that of Russia? to that of the West Indies? to that of Scotland? to that of Roumania? to that of New South Wales? to that of Hawaii?

13. Find the ratio of the area of Asia to the area of Africa; of the area of Africa to the area of North America; of the area of North America to the area of South America; of the area of South America to the area of Europe; of the area of Europe to the area of Oceania; of the area of Asia to the area of Europe; of the area of the United States to the area of Europe.

14. What would be the population of the United States if its density of population were that of Japan? of Asiatic Russia? of Hawaii? of Greenland? of Iceland? of Italy? of Greece? of Brazil? of Liberia? of Persia? of Scotland? of Nicaragua? of Switzerland? of Andorra? of Victoria? of Egypt?

15. What per cent is the area of England of the area of Europe? What per cent is the population of England of the population of Europe? Ascertain the same facts concerning

Norway.	Russia.	Spain.	Great Britain.
Sweden.	Switzerland.	Italy.	Austria Hungary.
France.	Greece.	San Marino.	Belgium.
Germany.	Servia.	Monaco.	Denmark.

16. Find the ratio of the area of Texas to the area of England; to the area of Egypt; to the area of France; to the area of Japan; to the area of Chile; to the area of Greenland; to the area of Mexico; to the area of Italy; to the area of Austria Hungary.

17. Find what per cent the area of New York is of the area of England; of the area of Uruguay; of the area of Greece; of the area of Cuba; ot the area of Belgium; of the area of Andorra.

18. Find the density of population of each of the divisions and countries whose density of population you have not already determined, and form a new table on the basis of the relative densities of population.

Problems in Common Fractions.

1. A rod is $16\frac{1}{2}$ feet, and a foot is 12 inches. How many inches are there in $15\frac{3}{8}$ rods?

2. How many rods are there in $546\frac{3}{8}$ inches?

3. A square rod is $272\frac{1}{4}$ square feet, and a square foot is 144 square inches. How many square inches are there in $6\frac{7}{8}$ square rods?

4. How many square rods are there in $96347\frac{1}{3}$ square inches?

5. What part of a square rod is $582\frac{1}{4}$ square inches?

6. If a barrel of kerosene containing 50 gallons is bought for $4.25 and sold at $0.12½ a gallon what is the gain?

7. The ages of three children are $6\frac{1}{2}$, $8\frac{1}{3}$, and $9\frac{7}{12}$ years respectively. What is their average age?

8. If cloth costs $\$\frac{3}{8}$ a yard how much can be bought for $\$3\frac{1}{3}$?

9. A certain field has 4 sides. The length of the first side is $25\frac{3}{8}$ rods, of the second $32\frac{7}{12}$ rods, of the third $31\frac{5}{16}$ rods, and of the fourth, $42\frac{3}{4}$ rods. What would it cost to fence the field at the rate of $37\frac{1}{2}$ cents a rod?

10 A farmer takes to a store $7\frac{1}{2}$ dozen eggs and $4\frac{3}{4}$ pounds of butter to exchange for flour at $5.25 per barrel. If the merchant gives 23 cents a dozen for eggs and 25 cents a pound for butter what is the balance due for a barrel of flour?

11. How many bushels of potatoes at $62\frac{1}{2}$ cents a bushel must be given for 18 yards of gingham at $12\frac{1}{2}$ cents a yard, and 9 pounds of sugar at $6\frac{1}{4}$ cents a pound?

12. A book-seller receives the following books: $\frac{1}{2}$ dozen grammars at $62\frac{1}{2}$ cents each, $\frac{3}{4}$ dozen spelling books at $37\frac{1}{2}$ cents each, and 5 arithmetics at $67\frac{1}{2}$ cents each. $\frac{1}{8}$ of the bill is deducted for prompt payment. What is paid for the books?

13. Three-sevenths of a certain man's property is $4895.60. What is his entire property?

[Represent the given part by a line 3 units long. A line how many units long will represent the entire property? What, then, is the ratio of the entire property to the given part?]

14. A man travels $\frac{5}{13}$ of a certain distance in $3\frac{23}{60}$ hours. How long will it take him to travel the remaining distance?

15. A man spends $\frac{3}{14}$ of his annual income for rent, $\frac{13}{24}$ for housekeeping expenses, $\frac{5}{84}$ for clothing, and $\frac{7}{60}$ for miscella-

neous expenses. He has saved at the end of the year \$66.20 What is his yearly income?

16. A laborer's yearly wages are a certain amount. One of his sons earns $\frac{5}{8}$ as much, and another $\frac{11}{12}$ as much. The three receive \$976. What are the wages of each?

[Suppose the wages of the father to be \$24, What, then, are the wages of the first son? of the second son? of the three? The real wages are how many times the supposed wages? What, then, are the real wages of the father? of the first son? of the second son?]

17. A man invests a certain capital, and at the end of each year increases his investment by the amount his wealth has increased during the year. The first year his profits are $\frac{1}{12}$ and his personal expenses $\frac{1}{18}$ his investment; the second year his profits are $\frac{2}{7}$ and his personal expenses $\frac{1}{7}$ of his increased investment; and the third year his profits are $\frac{3}{14}$ and his personal expenses $\frac{1}{8}$ of his second increased investment. At the beginning of the fourth year his investment is \$30500. What was his original investment?

18. A can do a certain piece of work in 12 days, and B can do the same work in 18 days. How many days will it take them both to do the work?

[What part of the work can A do in one day? What part can B do in one day? What part, then, can both do in one day? How many days, then, will it take them to do the work?]

19. A can do a piece of work in 36 minutes, B in 48 minutes, C in 42 minutes, and D in 54 minutes. In how many minutes can A and B do the work? A and C? A and D? B and C? B and D? C and D? A, B, and C? A, B, and D? B, C, and D? A, B, C, and D?

20. A cistern has 3 pipes. The first will fill it in 20 minutes, the second will fill it in 20 minutes, and the third will empty it in 60 minutes. In what time will it be filled with water running through the three pipes?

21. A can build a certain fence in 48 days, B in 54 days, C in 63 days, and D in 72 days. A and C work together four days. C then leaves the work, and B and E take his place. How many days will it take to complete the fence?

22. Would the value of the fraction $\frac{28}{35}$ be increased or diminished by adding 40 to both dividend and divisor?

Summary of Definitions.

Exact Divisor. A number contained without a remainder in a given number.

Common Divisor. A number contained without a remainder in two or more given numbers.

Greatest Common Divisor. The greatest number contained without a remainder in two or more given numbers.

Multiple. A number that will contain a given number without a remainder.

Common Multiple. A number that will contain two or more given numbers without a remainder.

Least Common Multiple. The least number that will contain two or more given numbers without a remainder.

Fraction. One or more equal parts of a unit.

An indicated division.

Decimal. A fraction in which the unit is divided into a number of parts denoted by a power of 10.

An expression of division in which the divisor is indicated by so placing a period, or decimal point, in the dividend that each figure at its right will represent a 0 of the divisor.

Common Fraction. A fraction in which the unit is divided into a number of parts not denoted by a power of 10.

An expression of division in which the dividend is written above and the divisor below a horizontal line.

Proper Fraction. A fraction in which the indicated operation cannot be performed.

Improper Fraction. A fraction in which the indicated operation can be performed.

Mixed Number. A whole number and a fraction considered as one number.

Reduction of Fractions. Changing the forms of fractions without changing their value.

Reciprocal of a Fraction. The fraction with its terms inverted.

Compound Fraction. A term applied to two or more fractions connected by the word 'of'.

Complex Fraction. A fraction with a fraction for a denominator, or numerator, or both.

Finite Decimal. The exact equivalent of some common fraction whose denominator contains no other factor than 2 or 5.

Infinite Decimal. The approximate equivalent of a common fraction whose denominator contains some other factor than 2 or 5.

Review Questions.

What numbers are divisible by 2? by 5? by 4? by 25? by 8? by 125? Explain the principle that governs the divisibility of a number by each of the preceding numbers.

What numbers are divisible by 3? by 9? by 11? Explain the principles governing such divisibility.

If a number is divisible by 5 and by 6, is it necessarily divisible by 30? Why?

If a number is divisible by 4 and by 6, is it necessarily divisible by 24? Why not?

Define a divisor; a common divisor; a greatest common divisor.

Find the greatest common divisor of 36, 48, and 63. Give a rule for finding the greatest common divisor of two or more numbers.

Find the least common multiple of 15 and 25; of 156 and 360. Give a rule for finding the least common multiple of two numbers.

Find the least common multiple of 36, 56, and 64; of 128, 320, 144, and 480. Give a rule for finding the least common multiple of more than two numbers.

Define cancellation. Reduce to its simplest form the expression $(48\times7\times5\times5\times150)\div(35\times3\times30\times8\times5)$. Give a rule for simplifying an expression by cancellation.

Define a fraction; a numerator; a denominator; a proper fraction; an improper fraction; a mixed number; a compound fraction; a complex fraction; a common denominator; a least common denominator; the reciprocal of a fraction; reduction of fractions. Give in written form an illustration of each.

Reduce $\frac{84}{128}$ to its lowest terms. Give a rule for reducing a fraction to its lowest terms.

$\frac{7}{12}+\frac{3}{12}=$? $\frac{7}{12}-\frac{3}{12}=$? Give a rule for adding fractions having a common denominator; for subtracting.

What fractions with a common denominator are equivalent to $\frac{2}{3}$ and $\frac{3}{4}$? Give a rule for reducing fractions to equivalent fractions having a common denominator.

$\frac{3}{4}+\frac{2}{3}=$? $\frac{3}{4}-\frac{2}{3}=$? Give a rule for adding fractions not having a common denominator; for subtracting.

$3\frac{5}{8}+4\frac{3}{7}=$? $4\frac{1}{5}-2\frac{3}{8}=$? Give a rule for adding mixed numbers; for subtracting.

Explain the multiplication of $\frac{5}{8}$ by 4; of $\frac{5}{8}$ by 3. Give a rule for multiplying a fraction by an integer.

Explain the multiplication of 8 by $\frac{3}{4}$; of 8 by $\frac{2}{3}$. Give a rule for multiplying an integer by a fraction.

Explain the multiplication of $\frac{2}{3}$ by $\frac{3}{4}$. Give a rule for multiplying a fraction by a fraction.

Explain the multiplication of $5\frac{3}{4}$ by 4; of 8 by $6\frac{3}{4}$. Give a rule for multiplying when the multiplier or the multiplicand is a mixed number.

Explain the division of $\frac{8}{9}$ by 4; of $\frac{8}{9}$ by 5. Give a rule for dividing a fraction by an integer.

Explain the division of 8 by $\frac{4}{5}$; of 8 by $\frac{5}{7}$. Give a rule for dividing an integer by a fraction.

Explain the division of $\frac{2}{3}$ by $\frac{3}{4}$. Give a rule for dividing when both dividend and divisor are fractions.

Explain the multiplication of $5\frac{2}{3}$ by $6\frac{3}{4}$. Give a rule for multiplying when both multiplier and multiplicand are mixed numbers.

Multiply $95\frac{1}{7}$ by $34\frac{1}{11}$. Give a rule for such multiplications.

Multiply $79\frac{1}{2}$ by $79\frac{1}{2}$; $9\frac{4}{11}$ by $9\frac{7}{11}$. Give a rule for such multiplications.

Multiply $24\frac{7}{12}$ by $15\frac{11}{12}$; $324\frac{2}{3}$ by $896\frac{1}{3}$. Give a rule for such multiplications.

Divide $2\frac{1}{2}$ by $3\frac{2}{3}$; 7 by $2\frac{2}{3}$. Give a rule for dividing by a mixed number.

Explain the division of $4\frac{2}{3}$ by 3. Give a rule for dividing a mixed number by an integer.

Explain the reduction of $2\frac{1}{2}$ to an improper fraction. Give a rule for reducing a mixed number to an improper fraction.

Explain the reduction of $\frac{17}{4}$ to a mixed number. Give a rule for reducing a mixed number to an improper fraction.

Write a compound fraction. Simplify it. Give a rule for simplifying a compound fraction.

Write a complex fraction. Simplify it. Give a rule for simplifying a complex fraction.

Can $\frac{3}{5}$ be reduced to an exact decimal? $\frac{7}{15}$? $\frac{12}{15}$? $\frac{7}{64}$? $\frac{27}{250}$? What fractions alone can be reduced to exact decimals? Reduce to a decimal $\frac{13}{25}$; $\frac{17}{32}$.

How many o's must be added to the numerator of $\frac{3}{4}$ that it may be changed to an exact decimal? to the numerator of $\frac{7}{125}$? to the numerator of $\frac{15}{16}$? to the numerator of any fraction whose denominaor contains only 2's and 5's.

What common fraction is eqnivalent to .5? to .05? to .25? to .005.? to .785? to .0005? to .0384? Give a rule for changing a decimal to a common fraction. Change each of the preceding decimals to a common fraction in its lowest terms.

Explain the multiplication of .075 by .34. Give a rule for multiplication of decimals.

Explain the division of 8.3925 by .45. Give a rule for division of decimals.

Explain the plan of paying for milk and cream at the Claremont Creamery.

Complete and read the table on pages 137 and 138.

Multiply, without writing the partial products, 964 by $3\frac{5}{8}$; 4967 by 43. Give a rule for such multiplications.

Explain in full the process of proving a multiplication by eliminating 9's; by eliminating 11's. Explain the process of proving a division by eliminating 9's; by eliminating 11's

Explain the obtaining of approximate results in multiplication of decimals; in division of decimals.

Explain the converting of factors into more convenient factors; the multiplying of 63 by 67, 54 by 56, etc.; the process of dividing in long division without writing the partial dividends.

Give the six Natural Divisions in the order of the area of each; of the population of each; of the density of population of each.

Explain each problem on page 153.

Review Exercises and Problems.

NOTE. Make all exercises of your own construction as difficult as possible, so as to thoroughly test your knowledge of the principles underlying them.

1. Write in figures a number of five integral and four decimal periods. Write the same number in words, writing the decimal portion first in separate periods and secondly by the common method.

2. Write in words, by both methods, a number of four integral and four decimal periods. Write the same number in figures.

3. Find the area of all the states west of the Mississippi River; find the population of these states. Find the area of the states east of the Mississippi River; find the population of these states. Find the density of population of the first group of states; of the second group. What would be the population of the first group if its density of population were that of the second? What would be the population of the second group if its density of population were that of the first?

4. Find the difference between the areas of each of the two following divisions: Hawaii and Connecticut; Italy aud Colorado; Belgium and New Hampshire; Philippine Islands and Arizona; Japan and California; Scotland and South Carolina; Texas and Austria-Hungary; England and New York.

5. Using multiplicands of ten figures, multiply by short methods by 25; by $3\frac{1}{3}$; by 125; by $16\frac{2}{3}$; by $333\frac{1}{3}$; by six other convenient parts of some power of ten.

6. Using a dividend of ten figures, divide by 250; by $12\frac{1}{2}$; by $1\frac{2}{3}$; by six other convenient parts of some power of ten. Express each result both as a complete quotient and as an integral quotient with a remainder.

7. Using multiplicands of six figures, multiply without performing any addition of partial products, by 56; by 72; by 64; by 240; by 360. Divide dividends of 8 figures by the same numbers, expressing each result both as a complete quotient and as an integral quotient with a remainder.

8. Using factors with 0's at their right, perform six exercises in multiplication. Using divisors ending with 0's, per form six exercises in division. Express each result both as a quotient and as an integral quotient with a remainder.

9. Using only two multipliers, multiply multiplicands of six figures by 568; by 497; by 246; by 832. Using only three multipliers, multiply multiplicands of six figures by 128328; by 756112; by 27963.

10. Without multiplying by any figure of the multiplier, multiply a multiplicand of six figures by 98; by 997; by 999; by 498; by 9975.

11. Without writing down either of the partial products, multiply by 11 a multiplicand of six figures; of eight figures; of fifteen figures.

12. Make and solve five exercises in the greatest common divisor; five in the least common multiple of two numbers; five in the least common multiple of more than two numbers.

13. Write five proper fractions and reduce them to their lowest terms. Write five improper fractions and reduce them to mixed numbers. Write five mixed numbers and re-

duce them to improper fractions. Make and solve five exercises in reducing fractions to equivalent fractions having a least common denominator.

14. Make and solve five exercises in addition of fractions; five in addition of mixed numbers; five in subtraction of fractions; five in subtraction of mixed numbers.

15. Make and solve five exercises in multiplying a fraction by an integer in which no cancellation can be performed; five exercises in which mental cancellation can be performed; five in which written cancellation will be performed.

16. Make and solve five exercises in each of the three similar classes in multiplying an integer by a fraction; in each of the three in multiplying a fraction by a fraction.

17. Make and solve five exercises in multiplying an integer by a mixed number; five in multiplying a mixed number by an integer; five in the multiplication of mixed numbers whose integral parts are comparatively small and whose fractional parts comparatively large; five in which the integral parts are comparatively large and the fractional parts comparatively small; five in which the denominators of the fractional parts are the same; five in which the integral parts are the same, and the sum of the fractional parts is 1.

18. Make and solve five exercises in dividing a fraction by an integer in which a divisor is contained in the dividend; five in which a divisor is not contained in a dividend, but in which cancellation can be performed; five in which no cancellation can be performed.

19. Make and solve fifteen exercises, classified like the preceding exercises, in dividing an integer by a fraction; fifteen, similarly classified, in dividing a fraction by a fraction; five in dividing a mixed number or a fraction by a mixed number; five in dividing an integer by a mixed number.

20. Make and solve five exercises in reducing compound fractions to simple fractions; five in reducing complex fractions to simple fractions.

21. Write five common fractions that can be reduced to finite decimals and perform the reduction; write five that cannot be reduced to finite decimals and extend each reduc-

tion through the repetend; write five decimals and reduce them to common fractions in their lowest terms.

22. Make and solve five exercises in multiplication of decimals in which each factor is a decimal; five in division of decimals in which only the dividend is a decimal; five in division of decimals in which both dividend and divisor are decimals, and in which the divisor does not contain more decimal places than the dividend; make and solve five similar exercises in which the divisor contains more decimal places than the dividend.

23. Multiply 2384 by $5\frac{1}{8}$ without writing the product by the five; make and solve four similar problems. Multiply 5349 by 43 without writing the product by the 4; make and solve four similar exercises.

24. Prove each of the multiplications in the preceding section by eliminating 9's; by eliminating 11's.

25. Multiply 93.485 by .064896, obtaining a product approximately correct to millionths; make and solve five similar exercises. Divide .00701654 by .095372, using such a number of figures in the dividend and the divisor that the product of the divisor and the quotient will approximately equal the dividend to ten-millionths; make and solve four similar exercises.

26. Find the product of 23, 3, and 24, using two more convenient factors in place of the 3 and the 24; make and solve four similar exercises.

27. Multiply 67 by 63 withont writing the partial products; make and solve four similar exercises.

28. Divide 6225 by 83 without writing the partial products; make and solve four similar exercises.

Compound Numbers and Mensuration.

What name do we give to the sharp end of a needle, or to any other extremity that has no appreciable length or breadth?

Draw the point of a needle lightly across some smooth substance as a smooth piece of wax.
What name do we give to its path as distinguished from a point?

Push a carpenter's plane several times across a rough board.
What name do we give to its path as distinguished from a line?

Place a heated piece of metal upon a block of ice, and allow it to melt its way through the block.
What name is given to its path as distinguished from a surface?

From the preceding inductive exercises we derive the following definitions:

A **Point** is that which has only position.

A **Line** is the path of a point in motion. It therefore extends, or "measures," in one direction, or, as the fact is commonly expressed, it has one dimension. This dimension is called **Length.**

A **Surface** is the path of a line in motion. It, therefore, extends, or "measures," in two directions, or, as the fact is commonly expressed, it has two dimensions. Its additional dimension, as compared with a line, is called **Breadth.**

A **Solid** is the path of a surface in motion. It, therefore, extends, or "measures," in three directions, or, as the fact is commonly expressed, it has three dimensions. Its additional dimension, as compared with a surface is called **Thickness.**

NOTE. The term width is used interchangeably with breadth. Thus, we speak of the width of a room, of a river, etc.

Extension toward the centre of the earth is referred to as depth, and extension from the centre of the earth as height. Thus, we speak of the depth of a mine, and the height of a mountain.

69. Measures of Length.

About how long is this book?
About how wide?
About how thick?

About how long is your school-room?
About how wide?

About how much ribbon, one inch wide and of medium quality, could you purchase for $1?

About how far is it around your school-house? [Give this distance in a larger denomination than has yet been used.]

About how far is it from your house to the nearest adjoining village?

In what denomination do we measure short distances like the dimensions of a book, of a pane of glass, etc.?
In what denomination the length of a room, of a table, etc.?
In what denomination the length of ribbons, dry-goods, etc.?
In what denomination distances like the dimensions of a field?
In what denomination long distances, like the distance between two villages, two towns, two cities, etc?

The relative values of the preceding denominations are expressed in the following

Table of Linear Measure.

12 inches (in.)	make	1 foot,	ft.
3 feet	"	1 yard,	yd.
$5\frac{1}{2}$ yards, or $16\frac{1}{2}$ feet,	"	1 rod,	rd.
320 rods	"	1 mile,	m.

NOTE 1. In races between running horses, the distances, which are a certain number of eighths of a mile, are referred to as so many FURLONGS. Thus, a race of $\frac{5}{8}$ of a mile is called a race of 5 furlongs, and a race of $1\frac{3}{8}$ miles, a race of 11 furlongs.

NOTE 2. In measuring the height of horses 4 inches is taken as a unit, and is called a HAND. Thus, a horse whose height is 5 feet 2 inches, or 62 inches, is spoken of as being $15\frac{1}{2}$ hands high. If the measurement is to be made within the fraction of an inch, it may be expressed in hands and inches. Thus, a horse may be said to be 15 hands $2\frac{1}{4}$ inches high, or $15\text{-}2\frac{1}{4}$.

NOTE 3. As will be explained under Circular Measure, every circumference is divided into 360 degrees, and each degree into 60 minutes. To 1 minute of the equatorial circumference of the earth the term GEOGRAPHICAL MILE is applied, this distance being a common unit among the mariners of all nations.

How many minutes are there in the circumference of any circle?

The equatorial circumference of the earth is 24899 statute miles. How many miles, then, are there in one minute of the equatorial circumference?

A geographical mile, then, is how many statute miles?

Observe that 1.15, the quotient of 24899 divided by 21600, is but a trifle less than $1\frac{1}{7}$. $1\frac{1}{7}$, therefore, may be remembered as the ratio of a geographical mile to a statute mile.

NOTE 4. The following special terms, also, are used at sea:

1. In recording the speed of vessels a geographical mile is termed a KNOT.

Thus the speed capacity of a vessel may be referred to as 19 knots an hour.

2. In referring to distances 3 geographical miles are termed a LEAGUE.

Thus, 27 geographical miles are referred to as 9 leagues.

3. In measuring depths 6 feet are called a FATHOM.

Thus, 300 feet are referred to as 50 fathoms.

NOTE 5. The official unit of length in the United States is a yard bar carefully preserved at Washington. This bar is composed of two metals, one of which expands under the influence of a change of temperature, while the other contracts to nearly an equal extent under the influence of the same change. As, moreover, the bar is kept in a room whose maximum change of temperature is but a fraction of a degree, it follows that the variation in the length of the bar can be but an insignificant quantity.

Ex. 86.

1. A horse in a certain race runs 9 furlongs in 1 minute 48 seconds. In what time, at the same rate, would it run a mile?

2. The height of a horse is 5 ft. 1 in., or 61 inches. What is its height expressed in hands?

3. The height of a horse is 5 ft. $3\frac{1}{4}$ in. What is its height expressed in hands?

4. The height of a horse expressed in hands is 14-3. What is its height in feet and inches?

5. The height of a horse expressed in hands is 16-$1\frac{1}{4}$. What is its height in feet and inches?

6. A ship sails a certain distance at the rate of $20\frac{1}{4}$ knots an hour. What is its rate in statute miles?

7. The United States battle-ship Maine on Oct. 17, 1894, made the following averages for 3-mile runs:

14.57 knots.	17.82 knots.
15.78 "	14.64 "
16.29 "	16.59 "
15.93 "	16.44 "

Find the equivalent in statute miles of each of the preceding averages.

8. By the treaty of 1867 the eastern boundary of Alaska in the vicinity of Mt. St. Elias is 10 leagues inland. The top of the mountain is 33.3 miles inland. Is the mountain in Alaska or in British America?

9. Which travels the faster, and how much—a train of cars moving at the rate of 27 miles an hour or a ship moving at the rate of $23\frac{1}{2}$ knots an hour?

10. The title of a certain book is "20000 Leagues Under the Sea." What would this distance be expressed in statute miles?

11. The distance between two islands is 937 leagues. What is the distance in common miles?

12. The depth of the sea at a certain point is 2384 fathoms. What is the depth expressed in feet?

70. Denominate and Compound Numbers.

The points a, b, c, d, and e lie in the same straight line. The distance from a to b is 3 rods, from b to c 2 yards, from c to d 1 foot, and from d to e 7 inches. What is the total distance from a to e?

Observe that the distances from a to b, b to c, etc., are expressed in different units, or **Denominations,** definitely fixed by law and bearing a definite relation to each other. Observe also that the total distance from a to e is represented by a single number composed of these four denominations.

Each of the four distances is called a **Denominate Number,** and the four taken together are called a **Compound Number.** Hence the following definitions:

A Denominate Number is a number consisting of units fixed by law and of definite magnitude.

A Compound Denominate Number is a denominate number composed of two or more definitely related denominations.

71. To Reduce a Compound Number to its Lowest Denomination.

We wish to reduce a number composed of yards, feet, and inches to inches.

How does the number of feet in a given distance compare with the number of yards in the same distance?

How, then, shall we reduce the given number of yards to feet?

With what shall we combine the product thus obtained?

How does the number of inches in a given distance compare with the number of feet in the same distance?

How, then, shall we reduce to inches the feet equivalent to the given number of yards and feet?

With what shall we combine the product thus obtained?

Give, then, a rule for reducing yards, feet, and inches to inches.

Give a rule for reducing a compound number of any denominations to units of the lowest denomination.

SOLUTION.

820
1640 rd.
9023 yd.
27071 ft.
324861 in.

We are to reduce 5 mi. 40 rd. 3 yd. 2 ft. 9 in. to inches.

EXPLANATION.

In 5 miles there will be 320 times as many rods as miles. We must first, therefore, multiply together 5 and 320. Multiplying, and combining the 40 rods with our product as we multiply, we find that 5 mi. 40 rd. equal 1640 rods.

In 1640 rods there are $5\frac{1}{2}$ times as many yards as rods. $\frac{1}{2}$ of 1640 is 820. We next multiply by 5, and as we multiply combine with our product the 820 yards and the 3 yards. We thus find that 1640 rd. 3 yd. equal 9023 yards.

Complete the explanation.

NOTE. Before solving the following exercises, review as fully as may be necessary Art. 63, and solve the first column of exercises in Ex. 81.

Ex. 87.

Reduce to units of their lowest denomination each of the following compound numbers:

1. 3 ft. 4 in.	5. 4 yd. 2 ft. 4 in.
2. 5 ft. 8 in.	6. 10 yd. 1 ft. 11 in.
3. 9 ft. 7 in.	7. 9 yd. 2 ft. 7 in.
4. 23 ft. 10 in.	8. 23 yd. 0 ft. 2 in.

9. **Reduce 34 rd. 4 yd. 2 ft. 9 in. to inches.**
10. **Reduce 47 mi. 289 rd. 2 ft. 8 in. to inches.**
11. **Reduce 86 mi. 76 rd. 3 yd. 2 ft. 6 in. to inches.**
12. **Reduce 82 mi. 35 rd. 2 yd. 2 ft. 4 in. to inches.**

72. To Reduce Units of a Lower Denomination to a Compound Number of Higher Denominations.

We wish to express a given number of inches as yards, feet, and inches.
How will the number of feet in a given distance compare with the number of inches in the same distance?
How, then, shall we reduce the number of inches to feet?
What shall we do with the remainder, if any?

How will the number of yards in a given distance compare with the number of feet in the same distance?
How, then, shall we reduce to yards the feet obtained by our first division?
What shall we do with our remainder, if any?
Give, then, a rule for reducing a given number of inches to a compound number composed of yards, feet, and inches.
Give a rule for reducing units of any denomination to a compound number of higher denominations.

Reduce 153467 inches to units of higher denominations.

SOLUTION.
153467 in.
12788-11 ft.
4262-2 yd.
8524
774-5 rd.
2-134 mi.

EXPLANATION.

In any distance there will be $\frac{1}{12}$ as many feet as inches. Our first step, therefore, is to divide the 153467 inches by 12. We thus find that 153467 inches equal 12788 ft. 11 in. We write the 11 inches as part of our answer, and reserve the 12788 feet to reduce to the next higher denomination.

In any distance there are $\frac{1}{3}$ as many yards as feet. Our second step, therefore, is to divide the 12788 feet by 3. We thus find that 12788 feet equal 4262 yd. 2 ft. We write the 2 feet as part of our answer, and retain the 4262 yards to reduce to the next higher denomination.

In any distance there are 1-$5\frac{1}{2}$, or $\frac{2}{11}$, as many rods as yards, Our third step, therefore, is to divide the 4262 yards by $5\frac{1}{2}$, or $\frac{11}{2}$. We thus find that 4262 yards equal 774 rd. 5 yd.

Complete the explanation.

NOTE. 1. For the finding of the true remainder when dividing twice 4262 by 11, instead of 4262 by $5\frac{1}{2}$, see Art. 35, Note 3.

NOTE. 2. When the remainder obtained by dividing by a mixed number is itself a mixed number, the fractional part of the remainder should be reduced to integers of lower denominations and combined with the corresponding units previously obtained. For an illustration of this process see the solution under Art. 75.

Ex. 88.

Reduce to units of higher denominations the following numbers.

1. 436728 in.	11. 97834 in.	21. 3859 yd.
2. 94329 in.	12. 246716 in.	22. 216784 in.
3. 116475 in.	13. 84316 ft.	23. 43496 yd.
4. 293438 in.	14. 9487 yd.	24. 937164 in.
5. 356208 in.	15. 23415 in.	25. 89324 yd.
6. 876432 in.	16. 3216 rd.	26. 217654 in.
7. 8735 in.	17. 82169 ft.	27. 97832 rd.
8. 29347 in.	18. 316728 in.	28. 843167 in.
9. 984316 in.	19. 9437 rd.	29. 916499 in.
10. 418216 in.	20. 34782 ft.	30. 84324 ft.

73. To Reduce a Fraction of a Higher Denomination to Integers of Lower Denominations.

We wish to reduce a fraction of a yard to integers of lower denominations.

How shall we reduce the given fraction of a yard to feet?

What shall we do with the integral part of the result?

What shall we do with the fractional part?

How shall we perform the reduction?

Give, then, a rule for reducing a fraction of a yard to feet and inches.

Give a rule for reducing a fraction of any higher denomination to integers of lower denominations.

Ex. 1. Reduce $\frac{5}{7}$ of a mile to lower denominations.

Ex. 2. Reduce .39 of a rod to integers of lower denominations.

SOLUTIONS.

(1)	(2)
$\frac{5}{7}$ mi.	.39 mi.
$\frac{1600}{7}$, $228\frac{4}{7}$, rd.	156
$\frac{22}{7}$, $3\frac{1}{7}$, yd.	124.8 rd.
$\frac{3}{7}$ ft.	4.4 yd.
$5\frac{1}{7}$ in.	1.2 ft.
	2.4 in.

EXPLANATIONS.

Ex. 1. $\frac{5}{7}$ mi. equals 320 times $\frac{5}{7}$, or $\frac{1600}{7}$, or $228\frac{4}{7}$, rods. The 228 rods we write as part of our answer; the $\frac{4}{7}$ rod we reserve to reduce to the next lower denomination.

$\frac{4}{7}$ rd. equals $5\frac{1}{2}$, or $\frac{11}{2}$, times $\frac{4}{7}$, or $\frac{22}{7}$, or $3\frac{1}{7}$, yards,
Complete the explanation.

Ex. 2. .39 mi. equals 320 times .39, or 124.8, rods. The 124 rods we write as part of our answer; the .8 rod we reserve to reduce to the next lower denomination.

Complete the explanation.

Ex. 89.

Reduce the following fractions to equivalent integers of lower denominations:

1. $\frac{7}{15}$ yd.	4. $\frac{8}{13}$ mi.	7. $\frac{7}{24}$ rd.	10. $\frac{5}{9}$ mi.
2. .17 mi.	5. .9 rd.	8. .43 mi.	11. .3 rd.
3. .83 rd.	6. .753 mi.	9. .36 mi.	12. .9 rd.

74. To Reduce a Compound Number to a Fraction of a Higher Denomination.

We wish to reduce a compound number composed of yards, feet, and inches to the fraction of a rod.

How shall we reduce the inches to a fraction of a foot?

With what shall we combine the fraction thus obtained?

How shall we reduce the mixed number representing the feet and inches to the fraction of a yard?

With what shall we combine the fraction thus obtained?

How shall we reduce the mixed number representing the yards, feet, and inches to the fraction of a rod?

Give, then, a rule for reducing yards, feet, and inches to the fraction of a rod.

Give a rule for reducing integers of any denomination to a fraction of a higher denomination.

Reduce 56 rd. 4 yd. 2 ft. 8 in. to a fraction of a mile.

SOLUTION.

$2\frac{2}{3}$ ft.
$4\frac{8}{9}$ yd.
$56\frac{8}{9}$ rd.

$$\frac{\overset{8}{\overset{32}{\cancel{512}}}}{9} \div \overset{5}{\overset{20}{\cancel{320}}}$$

$\frac{8}{45}$ mi.

EXPLANATION.

8 inches equal $\frac{2}{3}$ foot, and 2 ft. 8 in. equal $2\frac{2}{3}$ feet.

$2\frac{2}{3}$ feet equal $\frac{8}{3} \div 3$, or $\frac{8}{9}$, yard, and 4 yd. $2\frac{2}{3}$ ft. equal $4\frac{8}{9}$ yd.

$4\frac{8}{9}$ yards equal $\frac{44}{9} \div \frac{11}{2}$, or $\frac{8}{9}$, rod, and 56 rd. $4\frac{8}{9}$ yd. equal $56\frac{8}{9}$ rods.

$56\frac{8}{9}$ rods equal $\frac{512}{9} \div 320$, or $\frac{8}{45}$, mile.

Therefore, 56 rd. 4 yd. 2 ft. 8 in. equal $\frac{8}{45}$ mile.

Ex. 90.

1. Reduce 4 yd. 2 ft. 11 in. to the equivalent fraction of a rod.

2. Reduce 128 rd. 2 yd. 1 ft. 7 in. to the equivalent fraction of a mile.

3. Reduce 7 mi. 75 rd. 3 yd. 2 ft. 5 in. to miles and a fraction of a mile.

4. Reduce 95 rd. 2 ft. 3 in. to the equivalent fraction of a mile.

5. Reduce 175 rd. 4 yd. 10 in. to the equivalent fraction of a mile.

6. Make and solve the necessary number of additional exercises.

75. To Add Denominate Numbers.

We wish to add several compound numbers composed of yards, feet, and inches.

Explain in full the process of adding integral numbers of three orders.

What changes will you need to make in your explanation to make it apply to the addition of yards, feet, and inches?

Give, then, a rule for adding compound numbers composed of yards, feet, and inches.

Give a rule for adding compound numbers of any denominations.

We wish to add 5 mi. 14 rd. 1 yd. 2 ft. 5 in., 13 mi. 84 rd. 1 yd. 1 ft. 7 in., 23 mi. 116 rd. 2 yd. 2 ft. 5 in., and 48 mi. 73 rd. 1 yd. 1 ft. 11 in.

SOLUTION.

mi.	rd.	yd.	ft.	in.
	1	2	2	
5	14	1	2	5
13	84	1	1	7
23	116	2	2	5
48	73	1	1	11
89	288	$1\frac{1}{2}$	2	4
			1	6
89	288	2	0	10

EXPLANATION.

We first write the numbers so that units of the same order will fall in the same vertical column.

The sum of the column of inches is 28 inches, or 2 ft. 4 in. The 4 inches we write as part of our answer; the 2 feet we reserve to combine with the column of feet.

Proceeding in the same manner with the other columns, and combining with each column the units of the same order, if any, obtained from the preceding column, we find that the sum of the given numbers is 89 mi. 288 rd. $1\frac{1}{2}$ yd. 2 ft. 4 in.

To express the sum in its simplest form we reduce the $\frac{1}{2}$ yard to feet and inches, and combine the result with the feet and inches previously obtained. We thus obtain as the last three denominations of our sum 1 yd. 3 ft. 10 in., or, reducing our feet to yards, 2 yd. 0 ft. 10 in.

Ex. 91.

1. Add 23 mi. 39 rd. 5 yd. 2 ft. 6 in., 19 mi. 124 rd. 3 yd. 1 ft. 7 in., 25 mi. 76 rd. 4 ft. 6 in., 89 mi. 3 yd. 7 in., 8 mi. 54 rd. 1 yd. 1 ft. 8 in.

2. Add 2 yd. 5 in., 13 rd. 2 ft., 7 mi. 125 rd. 2 yd. 1 ft. 7 in., 20 mi. 9 in., 8 mi. 51 rd. 4 yd. 1 ft. 11 in.

3. Add 2 yd. 1 ft. 6 in., 2 ft. 9 in., 7 in., 14 rd. 5 yd., 87 rd. 4 yd. 2 ft. 6 in., 13 mi. 163 rd. 3 yd. 1 ft. 6 in.

4. Add 5 mi. 27 rd. 4 yd. 2 ft. 5 in., 27 mi. 3 yd. 10 in., 8 mi. 93 rd. 1 ft., 13 rd. 2 yd. 9 in.

5. Add 7 in. 2 ft. 5 in., 4 yd. 1 ft. 11., 39 rd. 4 yd. 2 ft. 8 in., 146 mi. 113 rd. 3 yd. 2 ft. 8 in.

6. Make and solve similar exercises until yon can solve them without error or hesitation.

76. To Subtract Denominate Numbers.

We wish to subtract a compound number composed of yards, feet, and inches from a larger number of the same denomination.

Explain in full the process of subtracting when both subtrahend and minuend are integral numbers of three orders.

What change will you need to make in your explanation to make it apply to subtraction when the subtrahend and the minuend are composed of yards, feet, and inches?

Give, then, a rule for subtracting when the subtrahend and the minuend are composed of yards, feet, and inches.

Give a rule for the subtraction of numbers of any denominations.

Subtract 5 mi. 133 rd. 5 yd. 1 ft. 6 in. from 13 mi. 64 rd. 2 yd. 2 ft. 10 in.

SOLUTION.

mi.	rd.	yd.	ft.	in.
13	64	2	2	10
5	133	5	1	6
7	250	$2\frac{1}{2}$	1	4
			1	6
7	250	2	2	10

EXPLANATION.

10 inches less 6 inches are 4 inches, and 2 feet less 1 foot are 1 foot.

We cannot subtract 5 yards from 2 yards. We therefore add $5\frac{1}{2}$ yards to our minuend and thus obtain $7\frac{1}{2}$ yards as a new min-

uend. From this minuend we subtract 5 yards and thus obtain 2½ yards as our remainder

To counterbalance the addition of the 5½ yards to the minuend we add one rod to the subtrahend. We thus obtain as a new subtrahend 134 rods.

Complete the explanation.

Ex. 92.

1. Subtract 3 mi. 40 rd. 4 yd. 2 ft. 10 in. from 8 mi. 90 rd. 1 yd. 2 ft. 11 in.

2. Subtract 17 mi. 5 yd. 2 ft. 7 in. from 25 mi. 124 rd. 4 yd. 2ft. 8 in.

3. Subtract 264 rd. 3 yd. 10 in. from 1 mi. 68 rd. 3 yd. 2 ft. 7 in.

4. Subtract 83 mi. 2 yd. 11 in. from 117 mi. 19 rd. 3 yd. 2 ft. 7 in.

4. Subtract 39 mi. 46 rd. 1 ft. 8 in. from 125 mi. 4 yd. 1 ft. 1 in.

6. Make and solve the necessary number of similar exercises.

77. To Multiply Denominate Numbers.

We wish to multiply by a given multiplier a compound number composed of yards, feet, and inches.

Explain in full the process of multiplying by a given multiplier an integral number of three orders.

What change would you need to make in your explanation to make it apply to the multiplication of a multiplicand composed of yards, feet, and inches?

Give a rule for the multiplication of a compound number of any denominations.

Multiply 3 mi. 164 rd. 1 yd. 2 ft. 11 in. by 8.

SOLUTION.

mi.	rd.	yd.	ft.	in.
4	2	7	7	
3	164	1	2	11
				8
28	34	4	2	4

EXPLANATION.

The product of 11 in. by 8 is 88 inches, or 7 ft. 4 in. The 4 inches we write as a part of the total product; the 7 feet we reserve to combine with the second partial product.

The product of 2 feet by 8 is 16 feet. Combining with these the 7 feet reserved from our first partial product we have 23 feet, or 7 yd. 2 ft. The 2 feet we write as a part of the total product; the 7 yards we reserve to combine with our third partial product.

Proceding in the same manner with the other denominations we find that our total product is 28 mi. 34 rd. 4 yd. 2 ft. 4 in.

NOTE. After multiplying the rods by 8 we must divide the product by 320. Multiplying by 8 and dividing by 320 is the same as dividing by 40. We therefore think of 164 as of 160 plus 4, and obtain the number of miles in our partial product by dividing 160 by 40. We next multiply the 4 rods by 8 and combine the product with the 2 rods reserved from the preceding product.

Ex. 93.

1. Multiply 5 mi. 48 rd. 3 yd. 2 ft. 7 in. by 9.
2. Multiply 37 mi. 123 rd. 4 yd. 10 in. by 5.
3. Mnltiply 36 mi. 234 rd. 4 yd. 2 ft. 7 in. by 125.
4. Multiply 40 mi. 2 yd. 8 in. by 289.
5. Multiply 413 mi. 316 rd. 4 yd. 2 ft. 11 in. by 999.
6. Make and solve the necessary number of similar exercises.

78. To Divide Denominate Numbers.

We are to divide by a given divisor a dividend composed of yards, feet, and inches.

Explain in full the process of dividing an integral number of three orders by any given divisor.

What change will you need to make in your explanation to make it apply to the division of a dividend composed of yards, feet, and inches?

Give, then, a rule for dividing a number composed of yards, feet, and inches.

Give a rule for the division of numbers of any denomination.

SOLUTION.

Divide 25 mi. 117 rd. 4 yd. 2 ft. 8 in. by 4.

mi	rd.	yd.	ft.	in.
25	117	4	2	8
6	109	2	1	9½

EXPLANATION.

25 miles are 24 miles plus 1 mile, and $\frac{1}{4}$ of 24 miles is 6 miles.

1 mile is 320 rods, and 320 rods plus 117 rods are 437 rods, or 436 rods plus 1 rod. $\frac{1}{4}$ of 436 rods is 109 rods. The second denomination of our quotient, therefore, is 109 rods.

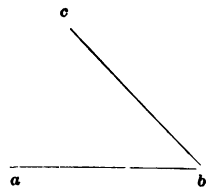
1 rod is $5\frac{1}{2}$ yards, and $5\frac{1}{2}$ yards plus 4 yards are $9\frac{1}{2}$ yards, or 8 yards plus $1\frac{1}{2}$ yards. $\frac{1}{4}$ of 8 yards is 2 yards. The third denomination of our quotient, therefore, is 2 yards.

Complete the explanation.

Ex. 94.

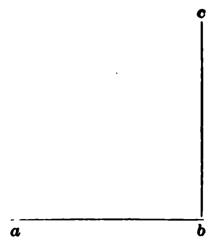
1. Divide 5 mi. 78 rd. 4 yd. 2 ft. 7 in. by 12.
2. Divide 37 mi. 3 yd. 1 ft. 8 in. by 25.
3. Make and solve the necessary number of similar exercises.

79. Angles and Surfaces.



Imagine the line *a b* in the accompanying diagram to revolve about the end *b* until it arrives at the position *b c*. Observe that there is considerable difference in the direction of the two lines. We state this fact in mathematical language by saying that the lines *a b* and *c b* form an **Angle** with each other. Hence the following definition:

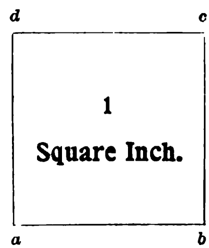
An Angle is the difference in direction of two lines that meet at a point.



Observe that the sides of the angle in the second diagram are so situated that the line *a b* in passing to the position *c b* has made one-fourth of a complete revolution. The angle thus produced is called a **Right Angle,** and the two sides are said to be **Perpendicular** to each other. Hence the following definitions:

A Right Angle is an angle in which the difference in direction is one-fourth of a revolution.

A Perpendicular is a line which meets another line at a right angle.



We have learned that the imaginary path of a line in motion is called a surface. Imagine a line an inch long to move perpendicularly to itself through an inch, as shown in the accompanying diagram. The surface thus passed over is called a **Square Inch.** Hence the following definition:

A Square Inch is the path of a line an inch in length moving perpendicularly to itself through an inch.

Naming other surfaces according to the same principle, what name do we give

To the path of a line a foot long moving perpendicularly to itself through a foot?

Of a line a yard long moving perpendicularly to itself through a yard?

Of a line a rod long moving perpendicularly to itself through a rod?

Of a line a mile long moving perpendicularly to itself through a mile?

Define, then,

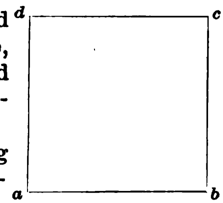
A square foot.	A square rod.
A square yard.	A square mile.

We may, therefore, give the following definition:

A **Unit of Surface** is the path of a Unit of Length moving perpendicularly to itself through a distance equal to its length.

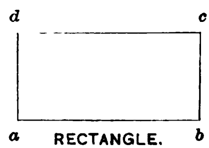
Each of the preceding surfaces, and every other surface of the same shape, whatever the length of the sides, is called a **Square**. Hence the following definition:

A Square is the path of any line moving perpendicularly to itself through a distance equal to its length.



SQUARE.

A line may move perpendicularly to itself, but instead of producing a square may produce a surface of the shape shown in the accompanying diagram; that is, a surface in which the producing line and the distance through which it moves are of unequal length. Such a surface is called a **Rectangle**.



RECTANGLE.

The term rectangle may, moreover, be used although the sides of the surface are of equal length. Hence the following definition;

A Rectangle is the path of a line moving perpendicularly to itself through any distance.

NOTE 1. To a rectangle with unequal sides the term **Oblong** is sometimes applied.

Note 2. The producing line of a rectangle, or the side on which the figure representing the rectangle is supposed to stand, is called the **Base.** A line representing the perpendicular distance through which the producing line moves, or a line perpendicular to the base, is called the **Altitude.**

Thus, in the rectangle *a b c d*, *a b* is the base, and *b c*, or *a d*, the altitude.

80. To Find the Area of a Rectangle.

Note. In each of the following problems the producing line is understood to move perpendicularly to itself.

What surface is produced by an inch moving through an inch?

How many square inches, then, will be produced
By 2 inches moving through an inch?
By 1 inch moving through 2 inches?
By 2 inches moving through 5 inches?
By 4 inches moving through 2 inches?
By 4 inches moving through 5 inches?
By 7 inches moving through 8 inches?
By 9 inches moving through 6 inches?
By 10 inches moving through 12 inches?
By 12 inches moving through 12 inches?

What surface is produced by a foot moving through a foot?
By 2 feet moving through a foot?
By 1 foot moving through 2 feet?
By 4 feet moving through 8 feet?
Ry 9 feet moving through 6 feet?
By 11 feet moving through 12 feet?
By 24 feet moving through $12\frac{1}{2}$ feet?

What surface is produced by a yard moving through a yard?
By 1 yard moving through 2 yards?
By 2 yards moving through 1 yard?
By 10 yards moving through 11 yards?
By $33\frac{1}{3}$ yards moving through 75 yards?

What surface is produced by a rod moving through a rod?
By 2 rods moving through 1 rod?
By 1 rod moving through 2 rods?
By 11 rods moving through 11 rods?
By 12 rods moving through $16\frac{1}{2}$ rods?

What surface is produced by a mile moving through a mile?
By 2 miles moving through 1 mile?
By 1 mile moving through 2 miles?
By 9 miles moving through 12 miles?
By 876 miles moving through 250 miles?

What process do we perform in each of the preceding exercises?
The length of the producing line forms what element of the operation?
The distance through which the producing line moves?
Give, then, a rule for finding the area of any rectangle.

For convenience of reference we here express the preceding rule:

To find the area of a rectangle, multiply the length of the producing line by the perpendicular distance through which it moves.

81. To Form a Table of Surface Measures.

A foot is equal to how many inches?
What surface is produced by a foot moving through a foot?
What surface is produced by 12 inches moving through 12 inches?
A square foot, then, is equal to how many square inches?

A yard is equal to how many feet?
What surface is produced by a yard moving through a yard?
What surface is produced by 3 feet moving through 3 feet?
A square yard, then, is equal to how many square feet?

A rod is equal to how many yards?
What surface is produced by a rod moving through a rod?
What surface is produced by $5\frac{1}{2}$ yards moving through $5\frac{1}{2}$ yards?
A square rod, then, is equal to how many square yards?

A rod is equal to how many feet?
What surface is produced by $16\frac{1}{2}$ feet moving through $16\frac{1}{2}$ feet?
A square rod, then, is equal to how many feet?

A mile is equal to how many rods?
What surface is produced by a mile moving through a mile?
What surface is produced by 320 rods moving through 320 rods?
A square mile, then, is equal to how many square rods?

One unit appears in Square Measure which has no corresponding unit in Long Measure. This unit is equal to 160 rods, and is called an Acre.
What expression composed of two factors represents the number of square rods in a square mile?
What expression, then, will represent the ratio of a square mile to an acre?
$(320 \times 320) \div 160 = ?$
How many acres, then, are there in a square mile?

For convenience of reference we express the relations that we have discovered between the different surface units in the following

Table of Square Measure.

12×12, or 144, square inches	make	1 square foot.
3×3, or 9, square feet	"	1 square yard.
$5\frac{1}{2} \times 5\frac{1}{2}$, or $30\frac{1}{4}$, square yards	"	1 square rod.
$16\frac{1}{2} \times 16\frac{1}{2}$, or $272\frac{1}{4}$, square feet	"	1 square rod.
320×320 square rods	"	1 square mile
160 square rods	"	1 acre.
$\frac{320 \times 320}{160}$, or 640, acres	"	1 square mile

In measuring land a metal chain is commonly used. This is made up of 100 links. The length of this chain, which forms the unit of surveyors' measure, is 4 rods.

How many rods are there in a chain?
How many square rods, then, are there in a square chain?
How many square rods are there in an acre?
What, then, is the ratio of an acre to a square chain?
How many square chains, then, are there in an acre?

Hence the following

Table of Surveyors' Square Measure.

4×4, or 16, square rods	make	1 square chain.
$160 \div 16$, or 10, square chains	"	1 acre.

NOTE 1. Except over very smooth surfaces, a half-chain is commonly used in place of the standard chain. All areas, however, should be expressed either in standard square chains or in acres.

NOTE 2. In railroad surveys, where length is the only dimension to be considered, a 100-foot metal tape is commonly used instead of the chain of 100 links. In measuring building lots, moreover, and all other small and valuable plots of land, the dimensions are commonly obtained in feet.

Ex. 95.

1. Add 43 sq. mi. 423 A. 97 sq. rd. 25 sq. yd. 8 sq. ft. 32 sq in., 84 sq. mi. 616 A. 148 sq. rd. 7 sq. ft. 38 sq. in., 31 sq. mi. 73 sq. rd. 7 sq. ft. 37 sq. in., 234 sq. mi. 315 A. 68 sq. rd. 19 sq. yd. 4 sq. ft. 117 sq. in.

2. Reduce $\frac{3}{17}$ mile to integers of lower denominations.

3. Subtract 72 sq. mi. 438 A. 108 sq. rd. 16 sq. yd. 5 sq. ft. 48 sq. in. from 98 sq. mi. 189 A. 75 sq. rd. 25 sq. yd. 2 sq. ft. 75 sq. in.

4. Reduce 234 A. 54 sq. rd. 23 sq. yd. 5 sq. ft. 29 sq. in. to the fraction of a mile.

5. Multiply 38 sq. mi. 42 A. 154 sq. rd. 20 sq. yd. 5 sq. ft. 131 sq. in. by 25.

6. Reduce 438376 square inches to units of higher denominations.

7. Reduce .37 square mile to integers of lower denominations.

9. Divide 217 sq. mi. 504 A. 107 sq. rd. 24 sq. yd. 3 sq. ft. 105 sq. in. by 17.

* * *

Ex. 1. A rectangular field is 75 rods long and 40 rods wide. What is its area?

Ex. 2. The base of a rectangle is 96 feet and the altitude $33\frac{1}{3}$ feet. What is its area?

Ex. 3. The area of a field is 120 square rods and its base 15 rods. What is its altitude?

SOLUTION.

(1)	(2)	(3)
3000	3200	8

EXPLANATION.

Ex. 1. We may think of the surface of this field as produced by a line 75 rods long moving perpendicularly to itself through 40 rods. The area of the field, therefore, is 40 times 75, or 3000, square rods.

Ex. 2. We may think of the base of this rectangle as the producing line, and of the altitude as the perpendicular distance through which the producing line moves. The area of the rectangle, therefore, is $33\frac{1}{3}$ times 96 or 3200, square feet.

Ex. 3. 15 rods, the producing line, or base, of the rectangle, in moving through 1 rod will produce 15 square rods. Therefore, to produce 120 square rods it must pass through 120 divided by 15, or 8 rods. The altitude of the rectangle, therefore, is 8 rods.

Ex. 96.

1. The boys of a certain school construct a scraper with which to remove the snow from the ice of a neighboring pond. The scraper clears a path $4\frac{1}{2}$ feet wide. If it is drawn $17\frac{3}{4}$ rods, how many square feet of surface will be cleared by it?

2. The producing line of a rectangle is 5 feet, and the distance through which it moves is $6\frac{3}{4}$ feet. What is its area?

3. The base of a rectangle is $37\frac{1}{2}$ feet, and its altitude is 40 feet. What is its area?

4. A certain field has four sides, and all its angles are right angles. One side is $72\frac{7}{25}$ rods, and an adjoining side is $27\frac{11}{25}$ rods. How many acres are there in the field?

5. The area of a rectangle is 936 square feet, and its altitude 48 square feet. What is its base?

6. One side of a square is 27 feet. What is the area?

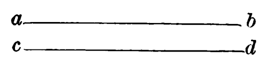
7. A rectangular field is 39.25 chains long and 23.18 chains wide. What is its area?

8. Select at your home or in your school-room ten small rectangular surfaces, such as the surfaces of sheets of paper, of panes of glass, etc. Obtain the dimensions of these surfaces to the nearest eighth of an inch and compute their areas.

9. Select ten large rectangular surfaces, such as the surfaces of doors, of the ends and sides of rooms, etc. Obtain the dimensions of these surfaces and compute their areas.

10. Select several rectangular land surfaces, such as yards, gardens, etc. Obtain the dimensions of these surfaces and compute their areas.

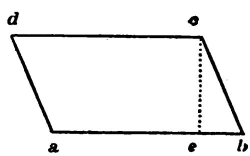
82. To Find the Area of a Parallelogram.



When two lines extend in the same direction they are said to be **Parallel** to each other. Thus, in the accompanying diagram the line *a b* is parallel to the line *c d*.

If a line moves parallel to itself will any surface be produced by its motion?

(Observe that while the representation of a line must have length, breadth, and thickness, a line itself is absolutely without breadth.)



PARALLELOGRAM.

A line which moves neither parallel to itself nor perpendicularly is said to move obliquely. Imagine the line *a b* to move obliquely till it arrives at the position *d c*. The resulting figure is called a **Parallelogram.**

Observe that the point *b* might have

arrived at the point *c* by moving first through the line *b e* and then through the line *e c*. In changing its position from *a b* to *d c*, then, through what distance parallel to the line *a b* does the point *b*, and every other point of the line *a b*, move?

Through what perpendicular distance does the line move?

Which alone of the preceding motions produces surface?

How, then, may we find the area of the parallelogram?

Give a rule for finding the area of any parallelogram.

How does this rule compare with the rule for finding the area of a rectangle?

What lines are measured in finding the area of a rectangle?

What lines in finding the area of a parallelogram that is not a rectangle?

NOTE 1. Any parallel-sided figure is a parallelogram. To an oblique-angled parallelogram the term RHOMBOID may be applied to distinguish it from a right-angled parallelogram, or rectangle. The term parallelogram, however, is commonly used as synonymous with rhomboid.

An equal-sided rhomboid is called a RHOMBUS. Observe that the rhombus has the same relation to the rhomboid that the square has to the rectangle with unequal sides.

NOTE 2. In measuring the altitude of a parallelogram be careful to so draw the line representing the altitude that the two angles formed by it with the base will be equal to each other.

Directions for erecting a perpendicular by geometrical principles will be given later.

SOLUTIONS.	
(1)	(2)
504	1080

Ex. 1. A certain field is in the form of a parallelogram. The length of each of the two longer sides is 42 rods and the perpendicular distance between these sides is 12 rods. What is the area of the field?

Ex. 2. The base of a parallelogram is 36 feet and the altitude 30 feet. What is the area of the parallelogram?

EXPLANATIONS.

Ex. 1. We may think of the surface of this parallelogram as equivalent to a surface produced by a line 42 rods long moving perpendicularly to itself through 12 rods. The area of the parallelogram, therefore, is 12 times 42, or 504, square rods.

Ex. 2. We may think of the base of this parallelogram as the producing line, and of the altitude as the perpendicular distance passed over by the producing line. The area of the parallelogram, therefore, is 30 times 36, or 1080, square feet.

Ex. 97.

1. The producing line of a parallelogram is 27 ft. 9 in. and the perpendicular distance through which it moves 8 ft. 10 in. What is the area in square feet?

2. The base of a parallelogram is 13 ft. 7 in. and its altitude 5 ft. What is its area?

3. A certain field is in the shape of a parallelogram. One side is 17.37 chains, and the perpendicular distance from this side to the opposite side is 12.25 chains. How many acres are there in the field?

4. The base of a rhomboid is 3 ft. 7 in. and the altitude 2 ft. 11 in. What is its area in square feet and square inches?

5. One side of a rhombus is 18 ft. 5 in. The perpendicular distance from this side to the opposite side is 11 ft. 4 in. What is its area in square feet?

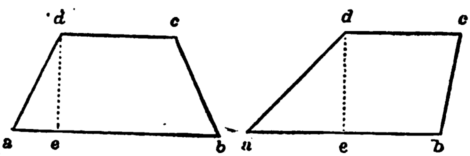
6. A city lot 100 feet front and 118 feet deep is sold at $1.75 per square foot. What is the value of the lot?

7. Draw ten rhomboids with bases varying from $1\frac{1}{4}$ to $3\frac{5}{8}$ inches, measure their altitudes, and find their areas.

8. Draw eight rhombuses, with the following bases: $1\frac{3}{8}$ inches, $2\frac{1}{4}$ inches, $4\frac{3}{8}$ inches, $3\frac{1}{2}$ inches, and $2\frac{7}{8}$ inches. Measure the altitude of each rhombus and find its area.

83. To Find the Area of a Trapezoid or a Triangle.

Imagine the line *a b* to move perpendicularly or obliquely to itself, and to constantly and regularly diminish in length. Imagine it to so move until it arrives at the position *d c*. The figure thus produced is called a **Trapezoid**.



TRAPEZOID. TRAPEZOID.

What line represents the producing line of each trapezoid in its original position?
In its final position?

A man earns $40 in June and $60 in July. What are his average monthly earnings?

A boy rides three hours on his bicycle. The first hour he rides 10 miles, the second hour 8 miles, and the third hour 12 miles. What is his average distance each hour?

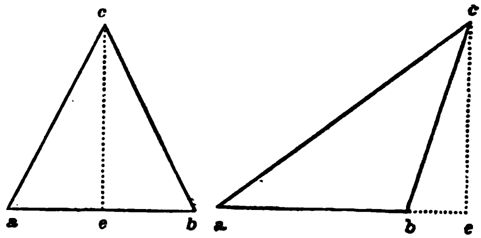
Following the principle applied in the preceding exercises, how shall we find the average length of the producing line of the trapezoid *a b c d*?

What line represents the perpendicular distance through which each producing line moves?

How, then, may we find the area of each trapezoid?

Give a rule for finding the area of any trapezoid.

Imagine the line *a b* to move perpendicularly or obliquely to itself, and to constantly and regularly diminish in length until it arrives at the point *c*. The figure thus produced is called a **Triangle**.



What line represents the producing line of each triangle in its original position?

What represents its final position?

What, then, will be its average length?

What line represents the distance through which each producing line moves?

How, then, can we find the area of each triangle?

Give a rule for finding the area of any triangle.

It is evident that the following definitions may be given of a trapezoid and a triangle:

A Triangle is the path of a line that constantly and regularly diminishes until reduced to a point.

A Trapezoid is the path of a line that constantly and regularly diminishes, but that ceases its motion before being reduced to a point.

All four-sided figures are called **Quadrilaterals.** The only quadrilateral not included in the preceding diagrams is the **Trapezium.** We may, therefore, combine all the preceding special rules into the following general rule.

To find the area of a triangle or of any quadrilateral not a trapezium multiply the average length of the producing line by the perpendicular distance through which it moves.

NOTE. The point where the two sides of an angle meet is called the **Vertex.** In a triangle that vertex which is opposite the base is

called the Vertex of the Triangle. To find the altitude ot a triangle measure the perpendicular from the vertex to the base or to the base extended.

The term base is by some writers applied to each of the parallel sides of a trapezoid, and to both the sides on which a parallelogram is supposed to stand and the side opposite that side.

	SOLUTION.	
	(1)	(2)
Ex. 1. The parallel sides of a trapezoid are 35 and 15 feet and its altitude 44 feet. What is its area?	1100	7200

Ex. 2. The base of a triangle is 150 feet and its altitude 96 feet. What is its area?

EXPLANATION.

Ex. 1. The longer of the parallel sides may be thought of as the producing line in its original position, and the shorter as the producing line in its final position. The average length of the producing line, therefore, is $(35+15)\div 2$, or 25, feet, and the area of the trapezoid is 25 times 44, or 1100, square feet.

Ex. 2. The base of the triangle may be thought of as the producing line in its original position, and the vertex as the producing line in its final position. The average length of the producing line, therefore, is $(150+0)\div 2$, or 75 feet, and the area of the triangle is 96 times 75, or 7200, square feet.

Ex. 98.

1. The producing line of a trapezoid moves perpendicularly to itself through 3 ft. 7 in. When it starts its length is 7 ft. 8 in., and when it stops its length is 5 ft. 8 in. What is the area of the surface produced by it?

2. The bases of a trapezoid are 17 ft. 7 in. and 12 ft. 5 in. respectively. Its altitude is 9 ft. 8 in. What is its area?

3. The producing line of a triangle is 7 ft. 9 in., and the distance through which it moves is 5 ft. 10 in. What is its area in square feet and square inches?

4. The base of a triangle is 27 rds. 11 ft., and its altitude 15 rds. 9 ft. 6 in. What is its area in square rods?

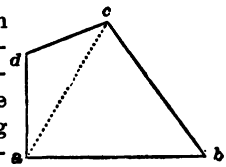
5. A field has three sides. The length of one side is 47.63 chains, and the length of a perpendicular let fall from the opposite corner upon this side is 38.72 chains. What is the area of the field expressed in acres?

6. A certain field has 4 sides, two of which are parallel to each other. The length of the longer parallel side is 43 rd. 8 ft., and of the shorter 29 rd. 5 ft. The perpendicular distance between the two sides is 34 rd. 7 ft. 6 in. What is the area of the field in acres?

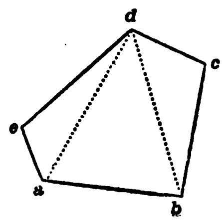
7. Cut from paper ten trapezoids of different dimensions and proportions. Measure their bases and altitudes and obtain their areas.

8. Cut from paper ten triangles. Measure their bases and altitudes and obtain their areas.

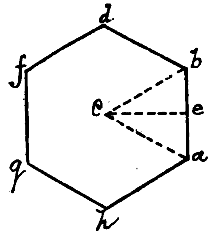
84. To Find the Area of a Trapezium or a Polygon.



The trapezium as shown in the accompanying diagram, is a quadrilateral in which no two sides are parallel. It follows, therefore, that the trapezium cannot be considered as produced by the progression of a straight line so moving that all positions are parallel to the original position. The process for finding the area of a trapezium and of all figures bounded by more than four straight lines is made evident by the accompanying diagrams.



To the second of the accompanying diagrams, and all other five-sided figures, is given the name **Pentagon**; to the third figure, and all other six-sided figures, the name **Hexagon**. In the same way, figures of seven sides are called **Heptagons**; of eight sides **Octagons**; of nine sides, **Nonagons**; of ten sides, **Decagons**.



From an examination of the accompanying diagrams give a general rule for finding the area of a trapezium or of any figure of more than four sides.

The name **Polygon** is a general term signifying a many-sided figure. The term, however, is generally restricted to figures of more than

four sides. A line connecting any two angles of a polygon not adjacent is called a Diagonal.

To the line or lines bounding a surface the term **Perimeter** is applied.

What are the perimeters of each of the preceding figures?

The last figure in the preceding diagrams, and every other polygon in which all the sides and angles are equal to each other, is called a **Regular Polygon.**

How will the sizes of the different triangles in the last figure compare with each other?

By what short method, then, can the area of a regular polygon be found?

Ex. 1. A field has the form of a trapezium. One diagonal is 60 rods, and the perpendicular distances to the diagonal from the opposite angles are respectively 48 and 24 rods. What is the area of the trapezium?

SOLUTION.

1440	34.6
720	120
2160	4152
$13\frac{1}{2}$ A.	

Ex. 2. A garden has the form of a regular hexagon. The length of each of its sides is 40 feet, and the perpendicular distance from the centre to each side is 34.6 feet. What is the area of the garden?

EXPLANATION.

Ex. 1. The given diagonal divides the trapezium into two triangles with a given base and altitude. Our problem, therefore, is simply to find the area of two triangles with a common base of 60 rods and with altitudes of 48 and 24 rods respectively. The area of these two triangles, and consequently the area of the trapezium, we find to be 1440 and 720, or 2160, square rods, or $13\frac{1}{2}$ acres.

Ex. 2. We may think of the hexagon as composed of six equal triangles, with a base of 40 feet and an altitude of 34.6 feet. The area of the hexagon, therefore, is $(40\times\frac{1}{2})\times 34.6\times 6$, or 34.6×120, or 4152, square feet.

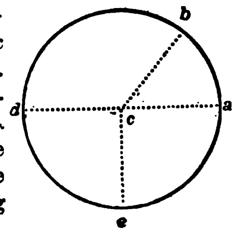
Ex. 99.

1. Draw five trapeziums, make the necessary measurements, and compute the areas.

2. Cut from paper the following figures: a nonagon, a pentagon, an octagon, a hexagon. Make the necessary measurements and compute the area of each figure.

85. To Find the Area of a Circle.

Imagine the line $c\ a$ in the accompanying diagram to revolve about the end c until it returns to its original position. The surface thus produced is called a **Circle**, the producing line is called the **Radius** of the circle, and the distance through which the free end moves, the **Circumference.** Hence the following definition:



CIRCLE.

A Circle is the surface produced by a straight line revolving about one of its ends.

What line represents the producing line in the preceding circle?
What represents the path of the end a?
Of the end c?
What, then, will be the average distance through which the points of the line $c\ a$ move?
How, then, may we find the area of the circle?
Give, then, a rule for finding the area of a circle.

For convenience of reference we here express the preceding rule.

To find the area of a circle multiply the length of the producing line, or the radius, by the average distance through which it moves, or 1-2 the circumference.

Instead of measuring the radius of a circle it is more convenient to measure the distance from a point in the circumference through the centre to the point directly opposite.

This distance, which is evidently twice the radius, is called the **Diameter.** The process of finding the diameter when the radius is given is self-evident.

It is, moreover, unnecessary to measure both the diameter and the circumference of a circle, as the circumference of every circle is 3.1416 times its diameter. To familiarize yourselves with the relations between these different elements, give the answers to the following inductive exercises:

What is the ratio of the diameter of a circle to its radius?
How, then, may we find the diameter of a circle when the radius is given?
How may we find a radius when a diameter is given?

What is the ratio of the circumference of a circle to its diameter?

How, then, may we find the circumference of a circle when a diameter is given?

How may we find the diameter when a circumference is given?

What is the ratio of the circumference of a circle to its radius?

How, then, may we find the circumference of a circle when the radius is given?

How may we find the radius when a circumference is given?

* * *

We have found that the area of a circle equals the radius multiplied by one half the circumference, or $r \times \frac{1}{2}c$. We wish to obtain an expression for the area of the circle in terms of the diameter.

In the expression $r \times \frac{1}{2}c$ what part of the diameter may we substitute for the radius?

How many times the diameter may we substitute for the circumference?

Making these substitutions, we have the expression $[(d \times \frac{1}{2}) \times (\frac{1}{2} \times d \times 3.1416)]$. Cancel 2 times 2 from the divisors, and 4 from the dividend of this expression. What factor remains in the dividend in place of 3.1416?

What expression may we use in place of 'diameter multiplied by diameter'?

What expression, then, represents the area of a circle in terms of the diameter?

Give a rule for finding the area of a circle in terms of the diameter.

* * *

We wish to find an expression for the area of a circle in terms of the circumference.

By what two divisors must we divide the circumference of a circle to obtain its radius?

In the expression $r \times \frac{1}{2}c$, then, what part of the circumference may be substituted for the radius?

Making this substitution, we have the expression $[(c \times \frac{1}{3.1416} \times \frac{1}{2}) \times \frac{1}{2}c]$.

What single factor may we substitute in the divisor in place of the factors 2 and 2?

What, then, are the two final factors of the divisor?

What expression may we use in place of 'circumference multiplied by circumference'?

What expression, then, represents the area of a circle in terms of the circumference?

Give a rule for finding the area of a circle in terms of the circumference.

Ex. 1. The diameter of a circle is 3½ inches. What is its area?

Ex. 2. The radius of a circle is 12 ft. 6 in. What is its area?

Ex. 3. The circumference of a circle is 18.8496. What is its area?

SOLUTIONS.

(1)	(2)	(3)
.7854	.78·54	18.8496
5.4978	19.63·5	9.4248
38.4846	490.875	28.2744
9.6212		

EXPLANATIONS.

Ex. 1. The area of a circle is the product of the producing line or radius, multiplied by the average distance through which it moves, or $\frac{1}{2}$ the circumference.

In place of the radius we substitute $\frac{1}{2}$ the diameter, and in place of $\frac{1}{2}$ the circumference $\frac{1}{2}$ the diameter multiplied by 3.1416. $\frac{1}{2}$ multiplied by $\frac{1}{2}$ is $\frac{1}{4}$, and $\frac{1}{4}$ of 3.1416 is .7854. We therefore find the area of the circle by multiplying .7854 by the diameter times the diameter, or by $\frac{7}{2}$ times $\frac{7}{2}$.

$\frac{7}{2}$ times $\frac{7}{2}$ are 7 times 7 times $\frac{1}{4}$. 7 times .7854 are 5.4978, 7 times 5.4978 are 38.4846, and $\frac{1}{4}$ of 38.4846 is 9.6212. The area of the circle, therefore, is 9.6212 square inches.

Ex. 2. The radius of the circle being 12 ft. 6 in., or $12\frac{1}{2}$ feet, the diameter is twice $12\frac{1}{2}$, or 25, feet.

Complete the explanation.

Ex. 3. The area of a circle equals the radius multiplied by $\frac{1}{2}$ the circumference.

In place of the radius we may substitute the circumference divided by 3.1416 and by 2. The area, therefore, is the circumference multiplied by the circumference, divided by 3.1416, 2 and 2.

The circumference is 18.8496 feet, therefore our dividend is composed of 18.8496 taken twice as a factor. We first divide one of these factors by 3.1416. The quotient thus obtained is 6. We complete the solution by multiplying the second factor of the dividend by $\frac{6}{4}$, or $\frac{3}{2}$. We thus find the area of the circle to be 28.2744 square inches.

NOTE 1. As a rule, in all problems involving only the processes of multiplication and division, indicate all the operations before performing any of them, This should be done both for the possible gain by cancellation and that a division by the product of all the factors of the divisor that cannot be cancelled may be the last step in the operation.

NOTE 2. Always look at the common sense of the answer to a problem, that is find the answer to a problem with elements approximately the same but so simple that they can be computed orally.

For example, in the first of the preceding exercises the area of a square enclosing the given circle would be about 12 square inches, and the area of the circle would be about $\frac{1}{4}$ less, or about 9 inches, In the same way, in the third exercise the diameter would be about 6 feet, the area of the square enclosing the circle about 36 square feet, and the area of the circle about $\frac{1}{4}$ less, or about 27 square feet.

How do the exact answers to these two exercises compare with these approximate answers?

Ex. 100.

1. Suppose a plank to be placed on edge on a strip of ice covered by a light snow, and one end of it to be fixed by a pin passing through it into the ice.

Supposing the length of the plank to be 20 feet, and thinking of a line forming one edge of the plank,

1. The distance passed over by its moving end while making a complete revolution will be how many times 3.1416?

2. The distance passed over by the middle point will be how many times 3.1416?

3. The average distance passed over by two points one 3 inches beyond and the other within 3 inches of the middle point will be how many times 3.1416?

4. The average distance passed over by all the points will be how many times 3.1416?

5. The surface cleared by the plank in making the revolution will be how many square feet?

2. The diameter of a circle is 42 rods. What is its area in acres and a fraction of an acre?

3. The radius of a circle is 56 rods. What is its area in acres and a decimal of an acre?

4. The circumference of a circle is 164 rods. What is its area in acres and square rods?

5. Find the ratio to four decimal places of the area of a circle with a diameter of 36 feet to the area of a square whose base is 36 feet?

Oral Problems.

1. The altitude of a square is 25 feet. What is its area?

2. The diameter of a circle is 10 feet. What is its area?

3. The base of a rectangle is 72 feet, and its altitude $33\frac{1}{3}$ feet. What is its area?

4. The base of a rhombus is 99 feet, and its altitude 75 feet. What is its area?

5. The radius of a circle is 25 feet. What is its area?

6. The base of a triangle is 96 feet, and its altitude 125 feet. What is its area?

7. The base of a rectangle is 24 feet, and its altitude 16 feet. What is its area?

8. The parallel sides of a trapezoid are 48 and 32 feet, and its altitude 28 feet. What is its area?

9. A diagonal of a trapezium is 18 feet, and the perpendicular distances to it from the opposite angles are 16 and 22 feet. What is its area?

10. The circumference of a circle is 12.5664 feet. What is its area?

11. The base of a rhomboid is $7\frac{3}{4}$ feet, and its altitude $7\frac{1}{4}$ feet. What is its area?

12. The four sides of a figure are equal and parallel. One side is 28 feet, and the perpendicular distance from this side to the opposite side is 25 feet. What is the figure and what is its area?

13. Two sides of a figure are parallel to each other. The length of the longer side is 36 feet and of the shorter 24 feet. The perpendicular distance between the two sides is 45 feet. What is the figure and what is its area?

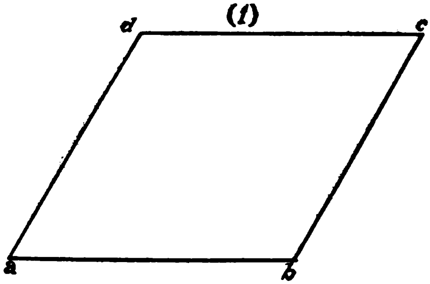
14. A figure has three sides. The length of one side is 25 feet, and the perpendicular distance to this side from the opposite angle is 36 feet. What is the figure and what is its area?

15. A figure has four equal sides, and all its angles are right angles. The length of one side is 35 feet. What is its area?

16. Name in order the figures on the following page, and give directions for finding the area of each.

17. Name and define all the figures previously given.

(1)



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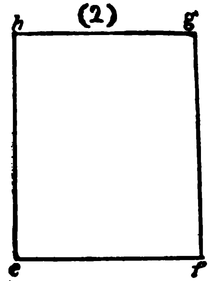

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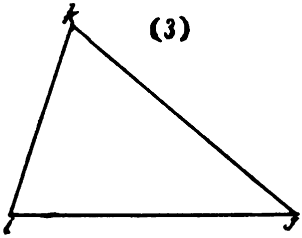

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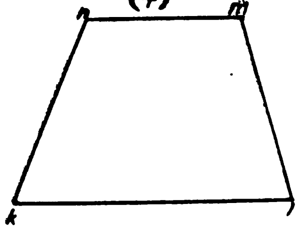

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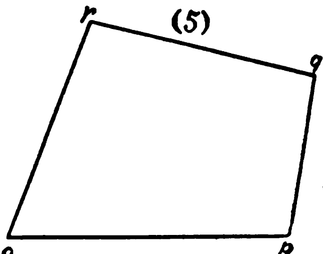

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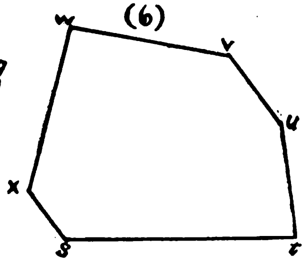

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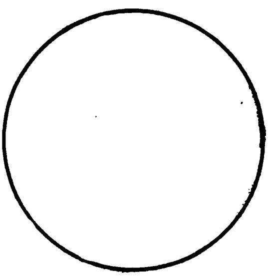

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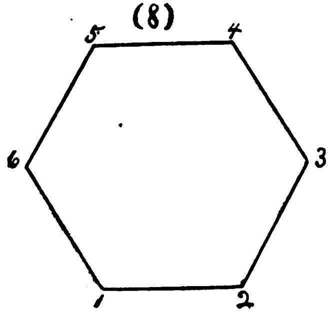


86. Prisms and Cylinders.

We have learned that the path of a surface is called a solid. Imagine a square inch to move perpendicularly to itself through an inch. The solid thus produced is called a **Cubic Inch.** Hence the following definition:

A Cubic Inch is the path of a square inch moving perpendicularly to itself through an inch.

Naming other solids according to the same principle, what name do we give

To the path of a square foot moving perpendicularly to itself through a foot?

To the path of a square yard moving perpendicularly to itself through a yard

To the path of a square rod moving perpendicularly to itself through a rod?

To the path of a square mile moving perpendicularly to itself through a mile?

Define, then,

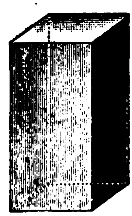
A cubic foot.	A cubic rod.
A cubic yard.	A cubic mile.

We may, evidently, give the following definition:

A unit of volume is the path of a unit of surface moving perpendicularly to itself through a distance equal to the length of one of its sides.

* * *

A square may move perpendicularly to itself through a distance not equal to the length of one of its sides, as shown in the accompanying diagram. To the resulting solid, and to the path of every polygon so moving that every position is parallel to the original position, is given the name **Prism.**



PRISM.

Prisms may be divided into two classes according to the direction of the motion of the producing polygon.

The producing polygon may move perpendicularly to itself. In this case the resulting prism is called a RIGHT PRISM.

The producing polygon may move obliquely. In this case the resulting prism is called an OBLIQUE PRISM.

Prisms are further classified according to the number of sides in the producing polygon. Thus,

If the producing surface is a triangle the resulting prism is called a TRIANGULAR PRISM.

If the producing surface is a quadrilateral the resulting prism is called a QUADRANGULAR PRISM.

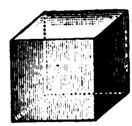
If the producing surface is a pentagon the resulting prism is called a PENTAGONAL PRISM.

Following the same principle, what name do we give
To a prism produced by a hexagon?
To a prism produced by a heptagon?
To a prism produced by an octagon?

Several divisions are made of quadrangular prisms. Thus,
If the producing quadrilateral is a parallelogram the resulting prism is called a **Parallelopiped,** or Parallelopipedon.

If the producing parallelogram is a rectangle so moving as to produce a right prism the resulting solid is called a RECTANGULAR PARALLELOPIPEDON.

If the producing rectangle is a square moving perpendicularly to itself through a distance equal to the length of one of its sides the resulting solid is called a **Cube.**


CUBE.

Prisms produced by regular polygons are appropriately called REGULAR PRISMS.

Observe also the following terms:

The first and the final position of the polygon producing a prism are called the ENDS of the prism.

The perpendicular distance through which the producing polygon moves is called the ALTITUDE of the prism.

The surfaces produced by the sides of the producing polygon are called the FACES of the prisms. The combined surfaces of the different faces are called the CONVEX SURFACE of the prism.

A circle may so move that every position will be parallel to its original position. The resulting solid is called a **Cylinder.** If the producing circle moves perpendicularly to itself the cylinder is called a RIGHT CYLINDER; if obliquely, an OBLIQUE CYLINDER.


CYLINDER.

NOTE. The terms END, ALTITUDE, and CONVEX SURFACE, are used in connection with the cylinder in the same signification as with the prism. The term

BASE, moreover, is used with both prisms and cylinders to signify the surface on which the figure is supposed to stand.

87. To Find the Volume of a Prism or a Cylinder.

The area of a polygon is 25 square feet. It moves perpendicularly to itself through 8 feet.
What is the volume of its path?

The area of the producing polygon of a prism is 25 square feet. The perpendicular distance through which it moves in producing the prism is 8 feet.
What is the volume of the resulting prism?

The area of the base of a prism is 25 square feet. The altitude is 8 feet.
What is its volume?

The area of a polygon is 25 square feet. It moves obliquely to itself, and moves through such a distance that the perpendicular distance between its original and its final position is 8 feet.
What is the volume of its path?

The area of the producing polygon of an oblique prism is 25 square feet. The perpendicular distance between its original and its final position is 8 feet.
What is the volume of the prism?

The area of the base of an oblique prism is 25 feet. The altitude is 8 feet.
What is its volume?

The area of a circle is 25 square feet. It moves perpendicularly to itself through 8 feet.
What is the volume of its path?

The area of the producing circle of a cylinder is 25 feet. The perpendicular distance through which it moves is 8 feet.
What is the volume of the resulting cylinder?

The area of the base of a right cylinder is 25 feet. Its altitude is 8 feet.
What is its volume?

The area of a circle is 25 feet. It moves obliquely through such a distance that the perpendicular distance between its original and its final position is 8 feet.
What is the volume of its path?

The area of the producing circle of an oblique cylinder is 25 square feet. The perpendicular distance between its original and its final position is 8 feet.
What is its volume?

The area of the base of an oblique cylinder is 25 feet. Its altitude is 8 feet.
What is its volume?

By what process is each of the preceding problems solved?
The area of the producing surface is what element of the operation?
The perpendicular distance through which the producing surface moves?
Give, then, a rule for finding the contents of any prism or cylinder.

For convenience of reference we here express the preceding rule.

To find the volume of a prism or a cylinder multiply the area of the producing surface by the perpendicular distance through which it moves.

Ex. 1. The adjacent sides of the base of a rectangular parallelopipedon are 15 and 18 feet, and the altitude of the parallelopipedon is 25 feet. What is its volume?

SOLUTIONS.

(1)	(2)
270	.7854
6750	471.24

Ex. 2. The diameter of the base of a cylinder is 5 feet, and the altitude of the cylinder is 24 feet. What is its volume?

EXPLANATIONS.

Ex. 1. The area of the base of the given parallelopipedon is 15 times 18, or 270, square feet. We may, therefore, think of the parallelopipedon as produced by a surface of 270 square feet moving perpendicularly to itself through 25 feet. Its volume, therefore, is 25 times 270, or 6750, cubic feet.

Ex. 2. The area of the base of the given cylinder is 5 times 5, or 25, times .7854 square feet. We may, therefore, think of the cylinder as produced by a surface of 25 times .7854 square feet moving perpendicularly to itself through 24 feet, and of the volume of the cylinder as 24 times 25 times .7854, or 471.24, cubic feet.

Ex. 101.

1. One side of the base of a triangular prism is 18 feet, and the perpendicular distance from this side to the opposite vertex is 12 feet. The altitude of the prism is 15 feet. What is its volume?

2. The diameter of the base of an oblique cylinder is 24 feet, and the altitude is 8 feet. What is its volume?

3. What is the volume of a cube one of whose edges is 23 feet?

4. What is the volume of a rectangular parallelopipedon the sides of whose bases are 27 and 35 feet, and whose altitude is 21 feet?

5. The circumference of the base of a cylinder is 7 inches, and the altitude of the cylinder is 14 inches. What is its volume?

6. Select five solids in the form of prisms, obtain their dimensions, and compute their volumes.

7. Select five solids in the form of cylinders, obtain their dimensions and compute their volumes.

88. To Form a Table of Solid Measure.

What volume is produced by a square foot moving through a foot?

A foot is how many inches?

What volume is produced by 12 times 12 square inches moving through 12 inches?

A cubic foot, then, is how many cubic inches?

Find in the same way the number

Of cubic feet in a cubic yard.

Of cubic yards in a cubic rod.

Of cubic rods in a cubic mile

Hence the following

Table of Cubic Measure.

$12 \times 12 \times 12$, or 1728, cubic inches	make	1 cubic foot, cu. ft.
$3 \times 3 \times 3$, or 27, cubic feet	"	1 cubic yard, cu. yd.
$5\frac{1}{2} \times 5\frac{1}{2} \times 5\frac{1}{2}$ cubic yards	"	1 cubic rod, cu. rd.
$320 \times 320 \times 320$ cubic rods	"	1 cubic mile, cu. mi.

Ex. 102.

1. Add 53 cu. yd. 16 cu. ft. 436 cu. in., 96 cu. yd. 1326 cu. in., 132 cu. yd. 18 cu. ft. 834 cu. in., 43 cu. yd. 16 cu. ft., 83 cu. yd. 9 cu. ft. 1720 cu. in.

2. Multiply 34 cu. yd. 13 cu. ft. 796 cu. in. by 125.

3. Reduce $\frac{5}{16}$ cubic yard to units of lower denominations.

4. Subtract 19 cu. yd. 16 cu. ft. 792 cu. in. from 26 cu. yd. 12 cu. ft. 359 cu. in.

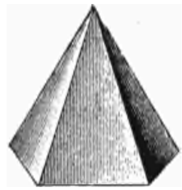
5. Reduce 21 cu. ft. 1342 cu. in. to a fraction of a cubic yard.

6. Divide 84 cu. yd. 14 cu. ft. 1638 cu. in. by 24.

7. Reduce 43 cu. yd. 14 cu. ft. 816 cu. in. to cubic inches.

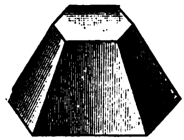
89. Pyramids and Cones.

A polygon may move obliquely or perpendicularly to itself and constantly and regularly diminish in surface until reduced to a point. The solid thus produced is called a **Pyramid.**



PYRAMID.

A polygon may move obliquely or perpendicularly to itself and constantly and regularly diminish in surface, but may cease its motion before being reduced to a point. The solid thus produced is called a **Frustrum** of a Pyramid.



FRUSTRUM OF PYRAMID.

What surface is produced by each side of the producing pologon when the solid is a pyramid? when the solid is a frustrum of a pyramid?

A circle may move obliquely or perpendicularly to itself and constantly and regularly diminish in surface uutil reduced to a point. The solid thus produced is called a **Cone.**

A circle may move obliquely or perpendicularly to itself and constantly and regularly diminish in surface, but may cease its mot ionbefore being reduced to a point. The solid thus produced in called a Frustrum of a Cone.

CONE.

NOTE 1. The basis of the classification of pyramids and cones is the same as that of prisms and cylinders. Thus, one speaks of Right pyramids and Oblique cones; of Triangular pyramids and Pentagonal cones.

FRUSTRUM OF CONE.

NOTE 2. The point at which the producing surface of a pyramid or a cone ceases its motion is called the APEX. The perpendicular distance from the apex to the perimeter of the producing surface is called the SLANT HEIGHT, and the perpendicular distance from the apex to the base the Altitude.

By a complex geometrical demonstration the following fact may be shown concerning the volume of a pyramid or a cone.

The volume of a pyramid or a cone is $\frac{1}{3}$ the volume of a prism or a cylinder with the same base and altitude.

SOLUTIONS.	
(1)	(2)
.7854	3360
314.16	
2513.28	

Ex. 1. The radius of the base of a cone is 10 feet, and the altitude of the cone is 24 feet. What is its volume?

Ex. 2. The base of a pyramid is a rectangle whose adjacent sides are 24 and 28 feet, and the altitude of the pyramid is 15 feet. What is its volume?

EXPLANATIONS.

Ex. 1. The area of the base of the cone is 20 times 20 times .7854 square feet, and the altitude of the cone is 24 feet. To find the volume of a cylinder with the preceding base and altitude we should multiply 20 times 20 times .7854 by 24. The volume of a cone is $\frac{1}{3}$ the volume of a cylinder with the same base and altitude. Therefore, the volume of a cone with the preceding base and altitude is $\frac{1}{3}$ of 24 times 20 times 20 times .7854, or 8 times 400 times .7854, or 2513.28, cubic feet.

Ex. 2. The volume of a prism with the same base and altitude would be 15 times 24 times 28 cubic feet, and the volume of the pyramid is $\frac{1}{3}$ of 15 times 24 times 28, or 5 times 24 times 28, or 120 times 28, or 3360, cubic feet.

Ex. 103.

1. Each side of the base of a regular hexagonal pyramid is 12 feet and the perpendicular distance from the centre of the base to each side is 10 feet $3\frac{1}{2}$ inches. The altitude of the pyramid is 15 inches. What is its volume?

2. The circumference of the base of a cone is 24 feet 4 inches, and the altitude of the cone 18 feet. What is its volume?

3. The base of a pyramid is a square each of whose sides is 9 feet, and the altitude of the pyramid is 7 feet. What is its volume?

4. The radius of the base of a cone is 5 feet, and the altitude of the cone is 11 feet. What is its volume?

5. The base of a pyramid is a rectangle whose adjacent sides are 22 and 26 feet, and the altitude of the pyramid is 19 feet. What is its volume?

6. The circumference of the base of a cone is 8 inches, and the altitude of the cone is 10 inches. What is its volume?

7. Find if possible five pyramids. Obtain their dimensions and compute their volumes.

8. Find if possible five cones. Obtain their dimensions and compute their volumes.

90. To Find the Volume of a Frustrum.

Suppose that a side of the frustrum of a pyramid represented on the preceding page is 25 feet in its original position, and 15 feet in its final position. Suppose also that the altitude of the frustrum is 18 feet.

How many feet has the line lost during its motion?
10 feet are what part of 25 feet?
What part of its original length has the line lost?
The distance through which the producing polygon has passed in producing the frustrum is, then, what part of the distance through which it would have passed had it continued its motion until reduced to a point?

The altitude of the pyramid, then, is what part of the altitude of the complete pyramid that would have been produced had the producing polygon not ceased its motion?

18 feet is $\frac{3}{5}$ of how many feet?
What, then, would have been the altitude of the complete pyramid?
How, then, can we find the volume of the complete pyramid?

The altitude of the frustrum of the pyramid is 18 feet, and we have found that the altitude of the complete pyramid would be 45 feet. What, then, would be the altitude of the pyramid that must be added to the frustrum to form with it the complete pyramid?
How, then, can we find the volume of the added pyramid?

Having given the volume of the added pyramid and the volume of the entire pyramid, how may the volume of the frustrum be found?

It is evident that the same principles would apply to finding the volume of the frustrum of a cone. Give, then, a rule for finding the volume of the frustrum of a cone.

Ex. 1. The dimensions of the larger end of the frustrum of a rectangular pyramid are 18 and 12 feet, and of the smaller end 12 and 8 feet. The altitude of the frustrum is 10 feet. What is its volume?

Ex. 2. Find the volume of the frustrum of a cone the diameters of whose ends are 24 and 10

SOLUTIONS.

(1)	(4)	
2160	.78·54	.7854
640	1178.10	6.2832
1520		50.2656
		452.3904
		5428.6848
		1178.10
		4250.5848

feet respectively, and whose altitude is 21 feet.

EXPLANATIONS.

Ex. 1. We first find the altitude of the pyramid that would have been produced had the producing rectangle not ceased its motion before being reduced to a point.

The longer side of the producing rectangle is 18 feet in its original position, and 12 feet in its final position. The side has been diminished, therefore, by 18 − 12, or 6, feet, or by $\frac{1}{3}$ its original length. The producing surface, therefore, has moved through $\frac{1}{3}$ the distance that it would have moved through in forming a complete pyramid. Therefore, 10 feet, the altitude of the frustrum of the pyramid, is $\frac{1}{3}$ the altitude of the complete imaginary pyramid, and the altitude of the complete pyramid is 3 times 10, or 30, feet. The volume of this pyramid, therefore, is $\frac{1}{3}$ of 30 times 18 times 12, or 10 times 216, or 2160, cubic feet.

We next find the volume of the pyramid that must be added to the frustrum to form the complete pyramid. The altitude is evidently 30 less 10, or 20, feet, and the volume is $\frac{1}{3}$ of 20 times 12 times 8, or 4 times 160, or 640, cubic feet.

The volume of the complete pyramid being 2160 cubic feet, and the volume of the added pyramid 640 cubic feet, the volume of the frustrum is 2160 less 640, or 1520, cubic feet.

Ex. 2. While the producing surface of the frustrum has moved perpendicularly to itself through 21 feet, its diameter has lost 14 feet, or $\frac{14}{24}$, or $\frac{7}{12}$, of its original length. Therefore, had the producing surface continued its motion until reduced to a point it would have passed through $\frac{12}{7}$ of 21 feet, or 36 feet.

We next find the volume of the complete imaginary cone. This volume is $\frac{1}{3}$ of 36 times 24 times 24 times .7854, or 12 times 3 times 8 times 3 times 8, or 12 times 9 times 8 times 8, times .7854, or 5428.6848, cubic feet.

The altitude of the imaginary added cone is 36 less 21, or 15, feet, and its volume is 15 times 10 times 10 times .7854, or 15 times 100 times .7854, or 1178.10, cubic feet. The volume of the frustrum, therefore, is 5428.6848 — 1178.10, or 4250.5848 cubic feet.

NOTE. Observe the four following distinct processes in the preceding solutions.

(1) To find the altitude of the complete pyramid or cone.
(2) To find the volume of this pyramid or cone.
(3) To find the volume of the added pyramid or cone.
(4) To find the volume of the frustrum.

Ex. 104.

1. A circle whose diameter is 3 feet moves perpendicularly to itself, and constantly and regularly diminishes until its diameter is reduced to 1 ft. 8 in. The perpendicular distance between its original and its final position is 3 ft. 6 in. What is the volume thus produced ?

2. The dimensions of the larger end of a frustrum of a rectangular pyramid are 24 and 16 feet, and the dimensions of the smaller end 18 and 12 feet. The altitude of the frustrum is 20 feet. What is its volume?

3. The ends of the frustrums of a right pyramid are squares. A side of the larger end is 17 inches, and of the smaller end 13 inches. The altitude of the frustrum is 16 inches. What is its volume?

4. The circumference of the larger end of a frustrum of a cone is 25 inches, and of the smaller end 9 inches. The altitude of the frustrum is 13 inches. What is its volume?

5. Find five frustrums of pyramids or cones, obtain their dimensions and compute their volumes.

91. To Find the Volume of a Sphere.

Suppose rectangular pyramids of the same base and altitude to be placed on a plane surface in contact with each other, and with each vertex lying in the same point. Suppose the bases to be so small that the part of the plane surface covered by the pyramids will be a polygon of a large number of sides and approximate a circle.

Suppose a second complete layer of the pyramids to be placed upon the first, and successive layers to be added, until the last layer consists of but a single pyramid. The solid thus produced will approximate in form the half of what single solid?

How will the altitude of each pyramid compare with the radius of the approximately equal half-sphere?

How will the combined surfaces of the bases of the pyramids compare with the surface of the half-sphere?

If a second solid similarly formed be placed in contact with the first it is evident that the two combined solids will approximate a sphere. It is also evident that the smaller the pyramids of which this solid is composed the more closely

the altitude of each pyramid will approximate the radius of the sphere to which the solid produced by the combined pyramids is constantly approximating and the more closely the combined bases of all the pyramids will approximate the surface of the sphere. Thinking, then, of a solid as composed of an infinite number of infinitely small pyramids, and considering the surface of the sphere to be a known quantity, give a rule for finding the volume of a sphere.

It is shown by geometry that the surface of a sphere equals its circumference multiplied by its diameter. We may, therefore, give the following rule:

To find the volume of a sphere multiply its surface, or its circumference multiplied by its diameter, by 1-3 its radius.

* * *

We have fouud that the volume of a sphere equals its circumference multiplied by its diameter multiplied by ⅓ its radius, or $c \times d \times \frac{1}{3} r$.

How many times the diameter may we substitute for the circumference?

What part of the diameter may we substitute for ⅓ the radius?

The new expression that we obtain for the volume of the sphere is $d \times 3.1416 \times d \times \frac{1}{6} d$. Cancel 6 from the 6 and the 3.1416. What factor remains in place of 3.1416?

What expression, then, represents the volume of a sphere in terms of its diameter?

Give a rule for finding the volume of a sphere in terms of its diameter.

NOTE. A sphere may also be thought of as the path of a circle revolving about a diameter through half of a complete revolution. Any producing circle in any position, that is any circle passing through the centre of the sphere, is callen a Great Circle of the sphere, and any radius or diameter of such a circle is called a radius or diameter of the sphere.

The diameter of a sphere is 25 inches. What is its volume?

SOLUTION.

3.1416
.52·36
13.09·
327.25·
8181.25

EXPLANATION.

We may think of the sphere as composed of an infinite number of infinitesimally small pyramids, the altitude of each of which is the radius of the sphere, and the combined surfaces of which are the surface of the sphere.

The surface of a sphere is shown by geometry to equal its circumference multiplied by its diameter. The volume of the pyramids, therefore, and the volume of the given sphere, is $(c \times d \times \frac{1}{3} r)$, or $(3.1416 \times d \times d \times \frac{1}{6} d)$, or $(.5236 \times d^3)$, or 8181.25, cubic inches.

Ex. 105.

1. The diameter of a sphere is 13 inches. What is its volume?

2. The radius of a sphere is 4 ft. 8 in. What is its volume in cubic feet and cubic inches?

3. The circumference of a sphere is 1 ft. $7\frac{1}{2}$ in. What is its volume?

4. Select five spheres and find their volumes.

92. Surfaces of Solids.

Except with the sphere, no new principle is involved in finding the area of the surfaces, or surface, that bound the preceding solids.

By what is the convex surface of a right prism or cylinder produced?
What name is given to the distance through which the perimeter of the base moves?
Give, then, a rule for finding the convex surface of a right prism or cylinder.

By what is the convex surface of a right pyramid or cone produced?
What is the length of the perimeter of the base in its final position?
What, then, will be its average length?
What name is given to the entire distance through which the producing perimeter moves?
Give, then, a rule for finding the convex surface of a right pyramid or cone.

By what is the convex surface of the frustrum of a right pyramid or cone produced?
How can the average length of the producing perimeter be found?
What name is given to the entire distance through which the producing perimeter moves?
Give, then, a rule for finding the convex surface of a frustrum of a right pyramid or cone.

Repeat the rule previously given for finding the surface of a sphere.

Ex. 1. The diameter of the base of a right cylinder is 6 feet, and the altitude of the cylinder is 5 feet. What is its convex surface?

SOLUTIONS.

(1)	(2)	(3)	(4)
3.1·416	$22\frac{1}{6}$	240	3.1416
94.248	$332\frac{1}{2}$		25.1328
			201.0624
			804.2496

Ex. 2. The adjacent sides of the bases of a rectangular pyramid are 9 ft. 6 in. and 12 ft. 8 in., and its slant height is 15 ft. What is its convex surface?

Ex. 3. The diameters of the bases of a frustrum of a right cone are 18 and 12 feet, and the slant height of the frustrum is 16 feet. What is its convex surface?

Ex. 4. What is the surface of a sphere whose diameter is 16 inches?

EXPLANATIONS.

Ex. 1. The convex surface of the cylinder is produced by the circumference of its base moving perpendicularly to itself through its altitude. It therefore is 5 times 3.1416 times 6, or 30 times 3.1416, or 94.248, square inches.

Ex. 2. The convex surface of the pyramid is produced by the constantly diminishing perimeter of its base moving perpendicularly to itself through its slant height. It therefore is 15 times $\frac{1}{2}$ of 2 times($9\frac{1}{2}$ plus $12\frac{2}{3}$,), or 15 times ($9\frac{1}{2}$ plus $12\frac{2}{3}$), or $332\frac{1}{2}$, square inches.

Ex. 3. The convex surface of the frustrum is produced by the constantly diminishing circumference of its base moving perpendicularly to itself through its slant height. It therefore is 16 times $\frac{1}{2}$ of (18 plus 12), or 16 times 15, or 240, square feet.

Ex. 4. It is shown by geometry that the surface of a sphere is the product of its circumference multiplied by its diameter. The surface of the given sphere, therefore, is 16 times 3.1416 times 16, or 2 times 8 times 2 times 8 times 3.1416, or 4 times 8 times 8 times 3.1416, or 804.2496, square inches.

Ex. 106.

1. The diameter of the base of a right cone is 25 feet, and the slant height of the cone is 40 feet. What is its convex surface?

2. The five sides of a right pentagonal prism are 8, 9, 12, 13, and 25 feet respectively. The altitude of the prism is 24 feet. What is its convex surface?

3. Each side of the base of a rectangular pyramid is 27 feet,

and the slant height of the pyramid is 13 ft. 10 in. What is its convex surface?

5. The diameter of the base of a right cylinder is 2 ft. 5 in. The altitude of the cylinder is 3 ft. 8 in. What is its total surface?

6. The diameter of a sphere is 1 ft. $2\frac{3}{8}$ in. What is its surface?

7. Select ten solids and find the total surface of each.

93. Similar Surfaces and Solids.

Two surfaces may have the same shape; that is, the longer dimension of the first surface may have the same ratio to the longer dimension of the second surface that the shorter dimension of the first surface has to the shorter dimension of the second surface. Such surfaces are called **Similar Surfaces**

The dimensions of a triangle are 26 and 39 inches. The lesser dimension of a similar triangle is 18 inches. What is its larger dimension?

The dimensions of a rectangle are 24 and 16 feet. The longer side of a similar rectangle is 30 feet. What is its shorter side?

The circumference of a given circle is 21 inches. The diameter of a second circle is 3 times the diameter of this circle. What is the circumference of the second circle?

Give the dimensions of two pairs of similar rectangles; of similar triangles; of similar rhomboids.

What would you say as to different circles being similar to each other?

Two triangles are similar. The base of the larger is 7 times the base of the smaller.

How, then, does the altitude of the larger compare with the altitude of the smaller?

How will the area of the larger compare with the area of the smaller?

The base of the larger of two similar rectangles is three times the base of the smaller.

How, then, does the altitude of the larger compare with the altitude of the smaller?

How will the area of the larger rectangle compare with the area of the smaller?

The radius of a given circle is 5 times the radius of another circle.

How, then, does the circumference of the larger compare with the circumference of the smaller?

How will the area of the larger compare with the area of the smaller?

The areas of two similar surfaces vary, then, as what power of their corresponding dimensions?

* * *

Two solids may have the same shape, that is, the greatest dimension of the larger may be to the greatest dimension of the smaller as each of the remaining dimensions of the larger is to the corresponding dimension of the smaller. Such solids are called **Similar Solids.**

The adjacent sides of the base of a rectangular prism are 40 and 35 inches, and the altitude of the prism is 50 inches. The altitude of a similar prism is 40 inches. What are its two other dimensions?

The altitude of a cone is 21 feet, and the diameter of its base is 14 feet. The altitude of a similar cone is 24 feet. What is the diameter of its base?

The diameter of a sphere is 8 inches. The circumference of a second sphere is 3 times that of the given sphere. What is the diameter of the second sphere?

Give the dimensions of two pairs of similar prisms; of similar pyramids; of similar cones. What would you say as to different spheres being similar to each other?

A side of the base of the larger of two similar prisms is 3 times the corresponding side of the smaller prism.

How, then, does the surface of the base of the larger prism compare with the surface of the base of the smaller prism?

How does the altitude of the larger prism compare with the altitude of the smaller?

How, then, will the volume of the larger prism compare with the volume of the smaller?

It is evident that the same principle will apply to all solids. The volumes of two similar solids vary, then, as what power of their corresponding dimensions?

SOLUTIONS.

(1)	(2)	(3)
15·6	675	$\$729^{0000}$
10·4	75	
	1875	

Ex 1. The dimensions of a rectangular parallelopipedon are 25, 24, and 16 inches. The longest dimension of a similar parallelopipedon is 16 ft. 3 in. What are the two other dimensions?

Ex. 2. The convex surface of a cone whose slant height is 27 inches is 675 square inches. What is the convex surface of a similar cone whose slant height is 45 inches?

Ex. 3. The value of a sphere of silver of a certain diameter is $1250. What would be the value of a sphere with a diameter 18 times as great?

EXPLANATIONS.

Ex. 1. By similar solids, we mean solids whose pairs of corresponding dimensions have a common ratio.

The ratio of the longest dimension of the second parallelopipedon to the corresponding dimension of the first is $16\frac{1}{4}$ divided by 25, or 65 divided by 100, or $\frac{13}{20}$.

$\frac{13}{20}$ of 24 equals $\frac{13}{10}$ of 12, or 15.6, and $\frac{13}{20}$ of 16 equals $\frac{13}{10}$ of 8, or 10.4. The required dimensions, therefore, are 15.6 and 10.4 inches.

Ex. 2. The slant height of the second similar cone is $\frac{45}{27}$, or $\frac{5}{3}$, that of the first. The circumference of the base of the second cone, or the producing line of its convex surface, is, therefore, $\frac{5}{3}$ that of the first, and the surface of the cone is $\frac{5}{3}$ times $\frac{5}{3}$, or $(\frac{5}{3})^2$, that of the first, or $\frac{25}{9}$ of 675 square inches, or 1875 square inches.

Ex. 3. The volume of a solid is the product of three dimensions, and all spheres are similar solids. The value of the second sphere, therefore, will be 18 times 18 times 18 times 1250, or 9 times 9 times 9 times 8 times 1250, or 7290000, dollars.

Ex. 107.

1. The value of a rectangular field is $250. The length of one of its sides is 46 rods. What is the value of a field of the same shape whose corresponding side is 90 rods?

2. A certain box in the form of a rectangular prism holds $7\frac{1}{2}$ bushels. How many bushels will a box of the same shape hold whose height is $\frac{1}{3}$ that of the first?

3. Two houses are of the same shape. The width of the first house is 30 feet and of the second house 24 feet. There are 2854 square feet in the sides and ends of the first house. How many square feet are there in the sides and ends of the second?

4. A log of a certain length that is 16 inches in diameter contains 165 board feet. How many board feet are there in a log of the same length that is 30 inches in diameter?

5. If $2.25 is paid for gilding a ball 1 ft. 8 in. in diameter, what would be the cost of gilding a ball 2 ft. 5 in. in diameter?

6. A pail whose larger end is $10\frac{1}{3}$ inches in diameter holds 11 quarts. What will a pail of the same shape hold whose larger end is $9\frac{1}{4}$ inches in diameter?

7. Find or make ten pairs of similar surfaces or solids, and determine their relative areas or volumes.

94. Practical Problems in Surface and Solid Measures.

1. Papering. Wall paper is manufactured in rolls 18 inches wide and 48 feet long. There is considerable waste in matching, which varies according to the design of the paper and the number of doors and windows in the room. This waste will be about $\frac{1}{6}$ of the paper purchased, and will be so estimated in the problems of this book.

How many square feet of paper are there in a roll?

A waste of $\frac{1}{6}$ being allowed for matching, how many feet of wall will be covered by a roll?

Give, then, a rule for finding the number of rolls of paper required for any given surface.

NOTE. As has been stated, paper is manufactured in rolls 48 feet long. Most paper-hangers, however, when using the term "roll" refer to a strip only 24 feet long, a strip 48 feet long being called by them a "double roll."

The term roll as used in this book will always refer to a strip of 48 feet.

SOLUTION.

66	89	$1.54
33	60	.77
561		$2.31
149		
412		

A room is 18 feet long and 15 feet wide. Its height from the base-board, which is 9 inches wide, is 9 ft. 9 in. It has 4 doors, each 7 feet by 3 ft. 6 in., and 3 windows, each 6 feet by 3 ft. 4 in. The width of the border is 18 inches. The price of the paper is 22 cents a roll, and of the border 3½ cents a linear yard. What will be the cost of the paper and border for the room?

EXPLANATION.

The perimeter of the room is twice 33, or 66, feet. The altitude of the space to be papered, reckoning to 3 inches above the border, is 9 ft. 9 in. less 15 in., or 8 ft. 6 in., and the surface to be papered, including the surface of the doors and windows, is $8\frac{1}{2}$ times 66, or 561, square feet.

The surface of the four doors, reckoning from the top of the base-boards, is 4 times $3\frac{1}{2}$ times $6\frac{1}{3}$, or 14 times $6\frac{1}{3}$, or, to the nearest integer, 89, square feet; and the surface of the three windows is 3 times 6 times $3\frac{1}{3}$, or 10 times 6, or 60, square feet. The combined surface of the doors and windows, therefore, is 89 plus 60, or 149, square feet, and the surface to be papered is 561 less 149, or 412, square feet.

60 square feet, the approximate surface covered by 1 roll, is con-

tained in 412 square feet nearly 7 times. The cost of the paper, therefore, will be 7 times 22 cents, or \$1.54. This amount combined with 22 times $3\frac{1}{2}$, or 77, cents, the cost of the border, gives as the total cost \$2.31.

Ex. 108.

1. A room is 14 ft. 4 in. long, 12 feet wide, and 8 ft. 4 in. high. It has two doors, each 7 ft. 6 in. by 3 ft. 6 in., and three windows, each 6 ft. 6 in. by 4 feet. The width of the baseboard is 8 inches, and the border is to be 9 inches wide. Find the quantity of paper and border required for the room.

2. Find the quantity of paper and border required for the walls and ceiling of your school room.

3. Select three rooms at your home and find the cost of suitable paper and border for each of them. Include paper and border for the ceiling of one of the three rooms.

* * *

2. Carpeting. Most carpeting is manufactured in rolls 1 yard wide, but the width of some of the higher grades is $\frac{3}{4}$ yard. It is sold by the linear yard. As with wall paper, there is a waste in matching. This waste can be definitely determined only with a knowledge of the length of the room and of the size of the figure in the design of the carpeting.

A breadth of ingrain carpeting, whose width is 1 yard, may be split in the middle. Full breadths of other carpeting must be purchased, however narrow the space that the last breadth is to cover.

NOTE. Unless otherwise stated the strips will be understood to run lengthwise of the room.

We wish to find the number of yards of carpeting 1 yard wide required for a certain room.
What will be the first step in the operation?
Suppose that the width of the room is not exactly divisible by 3 feet. What width of carpeting must be purchased for the last strip
If the remainder is $1\frac{1}{2}$ feet or less?
If the remainder is more than $1\frac{1}{2}$ feet?

To which of the strips will the waste in matching, if any, need be added?
What will be the final step in the operation?
Give, then, a rule for finding the number of yards of carpeting 1 yard wide required for a given room.
Modify this rule so it will apply to carpeting $\frac{3}{4}$ yard wide.

Ex. 1. What will be the cost at 60 cents a yard of carpeting for a room 18 feet long and 16 feet wide, the waste in matching being 6 inches?

Ex. 2. What will be the cost at 98 cents a yard of brussels carpeting, 27 inches wide, for the same room, the waste in matching being the same as with the first carpeting?

SOLUTIONS.

(1)	(2)
18	18
74	129½
9½	147.5^0
101.5	2 95
$20.30	144·55
	$48.18

Explanations.

The number of strips that must be purchased for the room will be the next integer greater than the quotient of 16 divided by 3, or 6.

The length of the first strip will be 18 feet; of the next 4 strips 4 times 18½, or 74, feet; and of the last strip, assuming that the waste in matching its second half to its first will be 6 inches, ½ of 18 feet, plus 6 inches, or 9½ feet. The total length of the carpeting, therefore, will be 18 plus 74 plus 9½, or 105.5, feet.

60 cents a yard is 20 cents a foot, therefore the cost of the carpeting will be .20, or $\frac{1}{5}$, of $101.5, or $20.30.

Ex. 2. The number of strips will be the next integer greater than the quotient of 16 divided by $\frac{9}{4}$, or 8.

The length of the first strip will be 18 feet, and of the remaining strips 7 times 18½, or 129½, feet. The total length of the carpeting, therefore, will be 18 plus 129½. or 147.5, feet.

The price of the carpeting is 98 cents a yard, or ⅓ of 98 cents a foot. We therefore multiply 147.5 by 98, or 100 less 2, and divide our product by 100 and by 3. We thus find the cost of the carpeting to be $48.18.

1. A room is 16 feet long and 14 feet wide. What will be the cost of ingrain carpeting for the room at 48 cents a yard if the waste in matching is 8 inches? What will be the cost of brussels carpeting at $1.32 a yard if the waste in matching is 4 inches?

2. Measure a room at your home. Find (1) the cost of brussels carpeting for the room at 85 cents a yard; (2) of ingrain carpeting at 54 cents; (3) of hemp carpeting at 20 cents. Assume a different waste in matching for each of the three kinds of carpeting. Find the cost of each kind of carpeting if the strips are made to run crosswise of the room.

3. Plastering. Mortar for ordinary plastering is made up of lime, sand, and hair in the proportion of 4 barrels of sand and 6 pounds of hair to 1 barrel of lime. With ordinary one-coat plastering the mortar from a barrel of lime will cover 50 square yards of surface.

A skilled mason will lay the mortar from about 1¼ barrels of lime in a day.

* *

In a certain house the surface to be plastered is 600 square yards.

How many barrels of lime will be required for the mortar? How many barrels of sand? How many pounds of hair?

How many days will it take a mason to do the plastering?

The price of the lime is $1.40 a barrel, and of the hair 6 cents a pound. The sand costs 25 cents a barrel and the slacking of the lime 50 cents a barrel. The mason is paid $3.50 a day, and his "helper" $1.75 a day.

What will be the cost of the lime? of the sand? of the hair?

What will the mason receive for his labor? What will the helper receive? What will be paid for slacking the lime?

Find the total cost of plastering the house; the cost a yard.

Ex. 109.

1. Find the cost at the preceding prices of plastering the room described in Ex. 1 on page 209.

2. Find the cost at the preceding prices of plastering your school room.

3. Find the cost at the prices current in your vicinity of plastering one or more of the rooms at your home.

4. Dealers in adamant wall plaster furnish and apply the material for 35 cents a square yard. Find the cost of plastering each of the preceding rooms with this plaster.

4. Painting. A gallon will cover 300 square feet—two coats—one fourth to one sixth of which will be the quantity required for trimming colors for windows, &c. No allowance should be made for the windows and other openings, as they about equal the extra surface of the frames, mouldings, &c.—*Extract from circular advertising ready mixed paints.*

A painter will paint in a day with a single coat about 90 square yards of outside surface, and about 50 square yards of inside surface. In these estimates the entire surface occupied by the windows should be included.

The outside surface of the walls of a house is 2100 square feet.

How many gallons of paint will be required for the house? What will be the cost of the paint at $1.50 a gallon?

In about how many days will a painter paint the house? What will be the cost of his labor at $2.50 a day? Find the total cost of painting the house.

Ex. 110.

1. Find the cost at the preceding prices of painting your school room and the outside of your school-house.

2. Find the cost of painting the house in which you live and of painting one or more of the rooms in it.

5. Brick Work. The regular dimensions of a brick are 8 by 4 by 2 inches. Bricks are laid on their sides, and lengthwise of the wall of which they form a part. They are laid in mortar, the thickness of which between the courses is such that 5 courses make approximately a foot. A barrel of lime and 4 barrels of sand will make mortar for laying about 1000 bricks.

* *

A chimney is 35 feet high, and its other two dimensions, exclusive of the mortar between the bricks, are 20 by 16 inches. We wish to find the number of bricks in the chimney.

The sum of the two sides and the two ends is how many inches? how many times 8 inches? A half of a corner brick at each corner counts as how many inches in both a side and an end? How many inches, then, will be gained at the 4 corners? How many bricks, then, less than 9 will be required for a course? how many for a course? How may we find the number of bricks that will be required for a course of any given dimensions?

How many courses of bricks are required to build a section of a chimney 1 foot high? How many courses, then, will there be in the entire chimney? how many bricks?

The price of the bricks is $9 a thousand. What will be the cost of the bricks for the chimney? of the mortar?

A mason can lay about 1000 bricks a day in a wall and about 600 in a chimney. Find the amount paid the mason for building the chimney; the amount paid his helper.

Ex. 111.

1. Find the approximate cost of a chimney 16 inches square and 32 feet high.

2. Find the approximate cost at the prevailing prices of building one of the chimneys in the house you live in.

6. Wood. When trees are made into wood they are frequently first cut into sticks 4 feet long. A pile of these sticks 8 feet long and 4 feet high makes an average load. Such a pile is taken as a unit of measurement and called a **Cord.** An eighth of a cord, moreover, is called a Cord Foot.

We wish to find the number of cords in a certain pile of wood.

How shall we find the number of cubic feet in the pile?

How shall we reduce the cubic feet to cords?

Give, then, a general rule for finding the number of cords of wood in a pile of any given dimensions.

* *

We wish to find the number of cords in a pile of four-foot wood of a given height and length.

What must be the product of the length and height of such a pile that the pile may be a cord?

How many cords, then, will there be in a pile of wood if the product of its length and height is

64 feet?	97 feet?	48 feet?
128 feet?	170 feet?	72 feet?

How many cords will there be in a pile of four-foot wood if its length and height are respectively

16 feet and 6 feet?	20 feet and 5 ft. 4 in.?
15 feet and 5 feet?	16 feet and 3 ft. 6 in.?
18 feet and 3 feet?	24 feet and 4 ft. 2 in.?

Give a special rule for finding the number of cords in a pile of four-foot wood of any given length and height.

* *

We wish to find the number of cords of four-foot wood in a pile four feet high and of any given length.

How long must such a pile be to contain one cord?

How many cords will there be, then, in a pile of four-foot wood that is four feet high and

24 feet long?	36 feet long?	41 ft. 8 in. long?
48 feet long?	72 feet long?	23 ft. 5 in. long?
54 feet long?	85 feet long?	37 ft. 7 in. long?

Give a special rule for finding the number of cords in a pile of four-foot wood that is 4 feet high and of any given length.

SOLUTIONS.

Ex. 1. A pile of four-foot wood is 28 feet long and $5\frac{1}{2}$ feet high. How many cords does it contain?

Ex. 2. A pile of four-foot wood is 4 feet high and 38 ft. 4 in. long. How many cords does it contain?

(1)

$38\frac{1}{2}$

$4\frac{13}{16}$

(2)

$4\frac{19}{24}$

(3)

$$\frac{109}{3} \times \frac{25}{6} \times \frac{7}{6} \times \frac{1}{128} = 1.38$$

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763          1536
19075       13824
13824
-----
 52510
 41472
 -----
 110380
```

Ex. 3. How many cords are there in 3 tiers of 14-inch wood, two of the tiers being 11 ft. 2 in. and the third 14 feet long, and the height of each tier being 4 ft. 2 in.

EXPLANATIONS.

Ex. 1. A pile of four-foot wood contains a cord if the product of its length by its height is 32. The number of cords in the given pile, therefore, is $5\frac{1}{2}$ times 28 divided by 32, or $5\frac{1}{2}$ times 7 divided by 8, or $4\frac{13}{16}$.

Ex. 2. A pile of four-foot wood 4 feet high and 8 feet long contains a cord. The number of cords in the given pile, therefore, is $38\frac{1}{3}$ divided by 8, or $4\frac{19}{24}$.

Ex. 3. The combined length of the 3 tiers is $36\frac{1}{3}$ feet. The number of cords in the pile, therefore, is $\frac{1}{128}$ of $\frac{109}{3}$ times $\frac{25}{6}$ times $\frac{7}{6}$, or 25 times 7 times 109 divided by 9 times 12 times 128, or 1.38.

Ex. 113.

1. A farmer on measuring a quantity of four-foot wood finds that it is in piles 4, 5, and 6 feet high. The combined length of the piles 4 feet high is 95 ft. 6 in., of the piles 5 feet high 84 feet, and of the piles 6 feet high 70 ft. 4 in. How many cords are there in all the piles?

2. The sled that the farmer uses in drawing the wood is 4 feet wide and 12 feet long. How high must the wood be piled that the sled may hold a cord?

3. A pile of four-foot wood is 5 ft. 6 in. high at one end, and its height regularly diminishes until it is reduced to 4 feet. If the length of the pile is 45 feet how many cords does it contain?

4. A pile of 12-inch wood consists of 7 tiers. The length of each tier is 18 feet. Three of the tiers are 15 ft. 6 in. high, three are 12 ft. 4 in. high, and one is only 10 feet high. How many cords does the pile contain?

5. A wagon body is 12 feet long and 4 feet wide. It is filled 3 feet high with tiers of 16-inch wood piled crosswise. Instead of holding 9 tiers, it is filled to the end by 8. What will the load measure if sold as 12 feet long? What will it measure if the length is considered to be 8 times the length of a stick?

6. A wood-shed is 16 feet high and 14 feet long. How wide must it be to hold 10 cords of four-foot wood cut into 16-inch

sticks, assuming that the space that would otherwise be required is increased 10 per cent. to allow for the space that will be wasted in piling the wood?

7. Find the number of cords in a pile of four-foot wood (*a*) 21 feet long and 5 feet high; (*b*) 37 ft. 5 in. long and 5 ft. 9 in. high; (*c*) 59 ft. 6 in. long and 4 ft. 6 in. high; (*d*) 14 ft. 5 in. long and 3 ft. 8 in. high; (*e*) 29 ft. 7 in. long and 4 feet high.

8. Find if possible five piles of wood, and measure them and obtain their contents.

* * *

7. **Lumber.** The unit of measurement for rectangular lumber, such as boards, planks, beams, etc., is a rectanglar solid with a base 1 foot square and an altitude of 1 inch. This unit is called a **Board Foot.** In the case, however, of a board less than an inch thick, a board foot signifies a rectangular solid with a base 1 foot square and an altitude equal to the thickness of the board, whatever that thickness may be.

A board of a certain length and breadth is 1 inch thick. How shall we find the number of board feet in the board?
How will the number of board feet in a plank of the same length and breadth but 3 inches thick compare with the number of feet in the given board?
Give, then, a general rule for finding the number of board feet in a board, plank, or other piece of rectangular lumber.
Modify this rule so as to make it apply to lumber less than an inch in thickness.

We wish to find the number of board feet in a plank 12 feet long $7\frac{1}{2}$ inches wide, and 3 inches thick.
What dimensions must we multiply together?
How many of these dimensions are in feet and how many in inches?
How many of the dimensions must be in feet and how many in inches that their product may express the contents of the plank in the desired denomination?

Suppose that we mentally indicate the product of the three dimensions without changing the denominations of any one of them. By what must we divide the resulting expression that it may represent the number of board feet in the plank?
What one of the three dimensions may be cancelled by this division?
The product of what two dimensions, then, would have given the number of board feet in the plank?

Suppose that instead of a plank we have a board 12 feet long and an inch or less in thickness. What one dimension will represent the number of board feet in the board?

What one dimension, then, expressed in inches, will represent the number of board feet in any board 12 feet long and 1 inch or less thick?

Give a special rule for finding the number of board feet in a piece of rectangular lumber 12 feet long and of any given width and thickness.

Give the number of board feet in a board 12 feet long, 1 inch or less thick, and

8 inches wide.	1 ft. 3 in. wide.
13 inches wide.	1 ft. 5 in. wide.
$9\frac{1}{2}$ inches wide.	2 ft. 1 in. wide.

Give the number of board feet in a piece of rectangular lumber 12 feet long whose breadth and thickness are respectively

10 and 2 inches.	13 and $3\frac{1}{2}$ inches.
$11\frac{1}{2}$ and 1 inch.	$8\frac{1}{2}$ and $6\frac{1}{2}$ inches.
17 and 3 inches.	$5\frac{3}{4}$ and $5\frac{1}{4}$ inches.
$10\frac{1}{2}$ and 4 inches.	21 and $2\frac{1}{2}$ inches.
$8\frac{1}{2}$ and 3 inches.	12 and $8\frac{1}{2}$ inches.

We wish to find the number of board feet in a plank $2\frac{1}{2}$ inches thick, 16 inches wide, and 15 feet long.

The product of what two dimensions would give the number of board feet in the plank if it were 12 feet long?

What is the ratio of 15 to 12?

How, then, after finding the number of board feet in the corresponding plank 12 feet long may we find the number of board feet in the given plank?

Give, then, a general rule for finding the number of board feet in a piece of rectangular lumber when its dimensions are expressed in feet, inches, and inches.

Give the number of board feet in a piece of rectangular lumber

16 feet long, 10 inches wide, and 3 inches thick.
18 feet long, $15\frac{1}{2}$ inches wide, and 2 inches thick.
24 feet long, 8 inches wide, and 6 inches thick.
8 feet long, 14 inches wide, and $\frac{3}{4}$ inch thick.
$13\frac{1}{2}$ feet long, 4 inches wide, and 4 inches thick.

* *

The waste occasioned by planing and matching lumber is paid for by the purchaser. Thus, while a half inch of the width of a four-inch board may be lost in planing and matching, the purchaser pays for the entire four inches.

The boards to be used for a certain floor have been reduced by planing and matching from 4 to $3\frac{1}{2}$ inches. We wish to find how many board feet must be purchased for the floor.

What fractional part of the rough boards is lost through planing and matching?
What part, only, then, can be used for the floor?
The surface of the floor, then, must have what ratio to the surface of the rough boards?
The surface of the rough boards, then, will have what ratio to the surface of the floor?
Give, then, a rule for finding the quantity of planed and matched boards required for a given surface.

Ex. 1. A plank is 12 feet long, $10\frac{1}{2}$ inches wide, and $2\frac{1}{2}$ inches thick. How many board feet does it contain?

SOLUTIONS.

(1)	(2)	(3)	
$26\frac{1}{4}$	475	180	\$2.160
		20	5.120
		640	\$7.28
		$213\frac{1}{3}$	

Ex. 2. How many board feet are there in 38 rafters, each 15 feet long, 5 inches wide, and 2 inches thick?

Ex. 3. The dimensions of a floor are 15 and 12 feet. The lining boards for the floor cost \$12 a thousand. The price of the flooring boards, which by planing and matching have been reduced in width from 4 to $3\frac{3}{8}$ inches, is \$24 a thousand. What will be the total cost of the material for the floor?

EXPLANATIONS.

Ex. 1. A board foot is the path of a square foot moving through an inch. The number of board feet in the given plank, therefore, is $2\frac{1}{2}$ times 12 times $\frac{1}{12}$ of $10\frac{1}{2}$, or $2\frac{1}{2}$ times $10\frac{1}{2}$, or 2 times 10 plus $\frac{1}{2}$ of 10 plus $\frac{1}{2}$ of 2 plus $\frac{1}{4}$, or $26\frac{1}{4}$.

Ex. 2. Each rafter contains $\frac{15}{12}$, or $\frac{5}{4}$, of 10, or $12\frac{1}{2}$, board feet, and the 38 rafters contain 38 times $12\frac{1}{2}$, or $\frac{1}{8}$ of 100 times 38, or 475, board feet.

Ex. 3. The surface of the floor is 12 times 15, or 180, square feet, and the cost of the lining boards will be .180 times \$12, or \$2.16.

$\frac{5}{32}$ of the width of the rough boards from which the flooring was manufactured was wasted in planing and matching. Therefore, the surface covered by the boards, or the surface of the floor, will be only $\frac{27}{32}$ of the surface paid for; the surface paid for will be $\frac{32}{27}$ of the surface of the floor, or $\frac{1}{3}$ of 32 times $\frac{1}{9}$ of 180, or $213\frac{1}{3}$, square feet; the cost of the boards will be $.213\frac{1}{3}$ times 24, or 5.12, dollars; and the cost of both the lining boards and the floor boards will be \$2.16 plus \$5.12, or \$7.28.

NOTE. The character × may be used between dimensions. Thus, we may speak of a plank 14 × 10 × 2, or 14 by 10 by 2.

Ex. 114.

1. Find the total contents in board feet of the following boards and planks:

150 boards 12 by 10, 173 boards 12 by 11, 207 boards 12 by 12, 119 boards 14 by 13, 156 boards 14 by 14, 215 boards 10 by 15, 335 boards 15 by 16, 408 planks 14 by 12 by $2\frac{1}{2}$, 243 planks 16 by 15 by $2\frac{1}{4}$, 135 planks 20 by 15 by 2.

2. Find the contents in board feet of the following sticks of rectangular lumber:

4 sticks $20\times6\times6$, 2 sticks $36\times6\times6$, 45 sticks $12\times6\times2$, 28 sticks $20\times6\times2$, 100 sticks $13\times2\times4$, 320 sticks $9\times2\times4$, 38 sticks $15\times2\times5$.

3. Find the cost at $40 a thousand of flooring for a room 20 by 18, the width of each board having been reduced by planing and matching from 3 to $2\frac{1}{2}$ inches.

4. Find the cost at $20 a thousand of flooring for your schoolroom. Assume (1) that the boards before being reduced by planing and matching were 6 inches wide; (2) that they were 4 inches wide; (3) that they were 3 inches wide.

* * *

8. **Clapboards.** Clapboards near large commercial centres are put up in bunches of 25 and sold by the thousand in number, each clapboard being 4 feet long and from 5 to 6 inches wide. In Vermont and New Hampshire, however, they are commonly sold by the thousand feet. In consequence of their being so laid as to considerably overlap each other, the width of the exposed surface covered by each clapboard is only from 3 to 4 inches.

We wish to find the number of board feet of clapboards required to be purchased for a given surface.

The clapboards used are 6 inches wide and are laid 4 inches to the weather. What is the ratio of the total surface of each clapboard to its exposed surface?

What, then, will be the ratio of the total surface of all the clapboards to the surface to be clapboarded?

How, then, shall we find the number of board feet of clapboards that must be purchased for the given surface?

Suppose that the surface is to be covered with clapboards 4 feet long laid a given number of inches to the weather.

How shall we find the surface that will be covered by a bundle of 25 clapboards?

Having found the surfacecovered by a bundle of clap-

boards, how shall we find the number of bundles required for a given surface?

Give, then, a general rule for finding the number of bundles of four-foot clapboards to be laid a given number of inches to the weather that will be required for a given surface.

SOLUTIONS.

(1)	(2)
36·00	108
432	\$75.60
\$61.71	

Ex. 1. The surface of the sides and ends of a building, not including doors, windows, etc., is 3600 feet. What will be the cost of clapboarding the building with clapboards 5 inches wide laid $3\frac{1}{2}$ inches to the weather, and costing \$12 a thousand feet.

Ex. 2. What will be the cost of clapboarding the preceding building at \$28 a thousand clapboards, each clapboard being laid 4 inches to the weather?

EXPLANATIONS.

Ex. 1. The exposed surface of each clapboard is $\frac{7}{10}$ of its entire surface. Therefore, the entire surface of the clapboards for the given surface is $\frac{10}{7}$ of the surface to be covered, or $\frac{10}{7}$ of 3600 square feet, and the cost of the clapboards is .001 of $\frac{10}{7}$ of 3600 times 12, or $\frac{1}{7}$ of 12 times .01 of 3600, or 61.71, dollars.

Ex. 2. A bundle of four-foot clapboards laid 4 inches to the weather will cover 25 times 4 times $\frac{1}{3}$, or $\frac{100}{3}$, square feet. The required number of bunches, therefore, is 3600 divided by $\frac{100}{3}$, or 108.

\$28 a thousand clapboards is equivalent to 70 cents a bundle. The cost of the clapboards, therefore, is 108 times .70, or 75.60, dollars.

Ex. 115.

1. How many board feet of 5-inch clapboards to be laid $3\frac{3}{4}$ inches to the weather must be purchased to cover a surface of 2500 square feet?

How many bundles of clapboards 4 feet long to be laid the same number of inches to the weather must be purchased for the same surface?

2. A house is 40 feet long and 30 feet wide. The height of its sides is 22 feet, and the perpendicular distance from the highest point of the gable to the base is 36 feet. It has three outside doors and 32 windows, the dimensions of the windows being 5 ft. 10 in. by 3 ft. 6 in., and of the doors 7 ft. 2 in. by 4

ft. 2 in. What will be the cost of clapboards for the house, the clapboards being 5 inches wide and being laid $3\frac{1}{4}$ inches to the weather, and costing $12.50 a thousand feet?

3. Find the number of board feet of clapboards, also the number of bundles of four-foot clapboards, required for your school-house. Find the quantity required for the house in which you live.

* * *

9. Shingles. Shingles are from 16 to 18 inches long. They are sold by the "thousand," this term signifying a quantity of shingles with a total width equal to the width of 1000 shingles each 4 inches wide. They are put up in bunches containing $\frac{1}{4}$ thousand each. Like clapboards they are made to overlap each other, and are commonly laid from 5 to 6 inches to the weather.

We wish to find how many thousands of shingles laid a given number of inches to the weather must be purchased for a given roof.

What is the width of the imaginary standard shingle?

The number of inches that it is to be laid to the weather being given, how shall we find the number of square inches that will be covered by it?

How shall we find the surface that will be covered by 1000 such shingles?

The surface that will be covered by a thousand of the imaginary standard shingles being given, how shall we find the number of thousands required for the given surface?

Give a general rule for finding the number of thousands of shingles required for any given surface?

A building is 36 feet long. Its rafters are 15 feet long, and the width of its jet is 14 inches. What will be the cost of shingling the roof with shingles laid $5\frac{3}{4}$ inches to the weather, $1 a thousand being paid for laying the shingles, and their price being $2.60 a thousand?

SOLUTION.

$$\frac{77}{2} \times 33 \times 144 \times \frac{1}{23} \times \frac{1}{1000} = 7.8$$

$28.80

```
   616
  5544
 16632
182·952
161
  219
```

EXPLANATIONS.

The total length of the surface to be covered is 36 feet plus twice 14 inches, or nearly $38\frac{1}{2}$ feet.

As the shingles should extend about $\frac{1}{2}$ inch beyond each end of the roof, the total length of the surface to be covered is 36 feet

plus 2 ft. 4 in. plus 1 inch, or nearly 38⅛ feet. The total width of the surface to be covered is twice (15 feet plus 1 ft. 2 in.), or 32 ft. 4 in., or, allowance being made for the two double layers and for these layers' extending over the jet, about 33 feet.

The surface covered by a thousand of shingles laid 5¾ inches to the weather is 1000 times 4 times 5¾ square inches, and the number of thousands of shingles required for the roof is 144 times 33 times $\frac{77}{2}$ divided by 1000 times 23, or $\frac{1}{23}$ of $\frac{1}{1000}$ of 11 times 3 times 9 times 8 times 77, or 7.8. 8 thousands of shingles, therefore, must be purchased, and the approximate cost of shingling the roof will be 8 times ($2.60 plus $1), or 8 times $3.60, or $28.80.

Ex. 116.

1. The surface of each side of a roof is 1000 square feet. What will be the cost of covering the roof with shingles at $2.50 a thousand and laid 6 inches to the weather?

2. Find the cost at $3 a thousand of shingles for a surface of 1832 square feet, the shingles being laid 5 inches to the weather.

3. A man about to re-shingle his house counts the courses of the old shingles and finds their number to be 60. The length of the roof is 42 feet. How many thousands of shingles must he purchase?

4. Find the quantity of shingles required for the roof of your school-house.

Find the quantity required for the roof of your house.

* * *

10. Laths. A lath is four feet long, and from 1 inch to 1⅜ inches wide. Laths are put up in bunches of 100 each. They are commonly laid about ¼ inch apart, this arrangement being equivalent to adding ¼ inch to the width of each lath.

We wish to find how many bunches of laths will be required for a certain room. The laths are of a given width and are laid a given distance apart.

How may we find the distance from the edge of any lath to the corresponding edge of the lath next to it?

How, then, may we find the distance that should be thought of as the width of each lath?

How shall we find the surface practically covered by each lath?

How shall we find the surface covered by a bunch of laths?

How shall we find the number of bunches required for the given surface?

Give a general rule for finding the number of bunches of laths required for any given surface.

An unfinished room is 10 ft. 6 in. wide, 11 feet long, and 8 feet high. The width of its base-board is to be 9 inches. It has two doors and two windows, its door frames being 7 ft. 1½ in. by 3 ft. 2 in., and its window frames, 5 ft. 3 in. by 2 ft. 10 in. The width of the laths is $1\frac{3}{8}$ inches, and they are laid ⅜ inch apart. They cost 25 cents a bunch, and 20 cents a bunch is paid for laying them. What will be the cost of lathing the room?

SOLUTION.

344	$12\frac{3}{4}$	$10\frac{1}{2}$
116	$3\frac{1}{6}$	$2\frac{5}{6}$
460	40	29
69		
391		

$$\overset{46.92}{391} \times \frac{1}{4} \times \frac{48}{7} \times \frac{1}{100} = \overset{\$3.15}{7}$$

EXPLANATION.

The surface of the walls, not including the doors and windows, is 8 times 43, or 344, square feet; the surface of the ceiling is $10\frac{1}{2}$ times 11 square feet, or, to the next higher integer, 116 square feet; and the total surface is 344 plus 116, or 460, square feet.

The surface of the two windows is 2 times $2\frac{5}{6}$ times $5\frac{1}{4}$, or $2\frac{5}{6}$ times $10\frac{1}{2}$, square feet, or, to the nearest integer, 29 square feet; the surface of the doors, reckoning from the top of the base-board, is 2 times $3\frac{1}{6}$ times $6\frac{3}{8}$, or $3\frac{1}{6}$ times $12\frac{3}{4}$, square feet, or, to the nearest integer, 40 square feet; and the combined surface of the doors and windows is 40 plus 29, or 69, square feet. The surface to be covered with laths, therefore, is 460 less 69, or 391, square feet.

The surface covered by a bunch of the laths is 100 times 4 times $\frac{7}{48}$ square feet. The required number of bunches, therefore, is 391 divided by 100 times 4 times $\frac{7}{48}$, or 7; and the approximate cost of lathing the room will be 7 times 45 cents, or $3.15.

NOTE. In speaking of the surface of a clapboard, lath, etc., reference is made only to the face which, in whole or in part, is to be laid to the weather.

Ex. 117.

1. Find the quantity of laths 1½ inches wide and laid $\frac{5}{16}$ inch apart required for a surface of 680 square feet.

2. Find the cost of lathing the room described in exercise 1, page 209, the cost of the laths being 30 cents a bunch, and the price of laying them 25 cents a bunch.

3. Find the cost at 28 cents a bunch of lath for your school room; of lath for your house.

95. United States Money.

That which is intended by a Nation to serve as a general medium of exchange among its people is called Money. The classes of United States Money now issued (1902) are shown by the following diagram.

- United States Money
  - Coins
    - Gold
      - Double Eagle, Unlimited Legal Tender.
      - Eagle ($10) Unlimited Legal Tender.
      - Half Eagle, Unlimited Legal Tender.
      - Quarter Eagle, Unlimited Legal Tender.
    - Silver
      - Dollar, Unlimited Legal Tender.
      - Half Dollar, Legal Tender for $10.
      - Quarter Dollar, Legal Tender for $10.
      - Dime (10 cts.), Legal Tender for $10.
    - Nickel
      - Five Cents, Legal Tender for 25 cents.
    - Bronze
      - One Cent, Legal Tender for 25 cents.
  - Paper Money
    - Issued by Government
      - Legal Tender
        - Treasury Note.
        - United States Note.
      - Not Legal Tender
        - Gold Certificate.
        - Silver Certificate.
    - Issued by Banks
      - Bank Notes, Not Legal Tender.

NOTE 1. By legal tender is signified such money as may be legally offered in payment of a debt, or in exchange for a privilege or a commodity. Thus, the ticket agent of a railroad can refuse to give a ticket in exchange for a bank note or a gold certificate. If, however, he refuses to accept notes of the issue of 1863 or other unlimited legal tender money, or any of the minor silver coins if the cost of the ticket is not more than $10, the road is responsible for any inconvenience or financial loss thus occasioned.

NOTE 2. The original unit of value in the United States was the silver dollar. This was adopted on April 2, 1792, and was made to contain $371\frac{1}{4}$ grains of pure silver. By the same act the ratio of gold to silver was placed at 15 to 1, the gold eagle being made to contain 247.5 grains. In 1834 the weight of pure gold in the eagle was reduced to 232 grains, and in 1837 it was increased from that amount to 232.2 grains.

What was the ratio of the weight of the silver in the silver dollar to the weight of the gold in the gold dollar from 1834 to 1837? What is the present ratio?

NOTE 3. The substance of the matter on the face and the back of each kind of paper money is shown in the following diagrams:

UNITED STATES NOTE. (Face)

Act of March 3, 1863.

This Note is a Legal Tender
For One Dollar.
The United States
Will Pay to Bearer
One Dollar
Washington, D. C.
United States Note.

(Back)

This Note is Legal Tender at its Face Value for all Debts, Public and Private, except Duties on Imports and Interest on the Public Debt.

BANK NOTE. (Face)

National Currency
Secured by United States Bonds
Deposited with the Treasurer of
The United States
The National Bank
Of White River Junction
Will pay the bearer on demand
Five Dollars
White River Junction, Vermont
April 9, 1886.

(Back)

This Note is receivable at par in all parts of the United States in payment of all Taxes and Excises and all other Dues to the United States except duties on Imports. Also for all Salaries and other Debts and Demands owing by the United States to Individuals, Corporations, and Associations within the United States except Interest on the Public Debt.

GOLD CERTIFICATE. (Face)

Act of July 12, 1882.
This certifies that there have been deposited in the Treasury
of the
United States
One Hundred Dollars in
Gold Coin
repayable to the bearer on
demand.

(Back)

United States
Gold Certificate.

SILVER CERTIFICATE. (Face)

Act. of Aug. 4, 1886.

This certifies that there has
been deposited in the treasury
Of the United States
One Silver Dollar
Payable to bearer on demand
Washington, D. C.

(Back)

United States
One
This Certificate is Receivable for Customs, Taxes, and all Public Dues, and when so Received
may be Reissued
Dollar
Silver Certificate

TREASURY NOTE. (Face)

Legal Tender Act, July 14, 1890.
The United States of America
Will pay to Bearer
Twenty Dollars
In Coin
Treasury Note

(Back)

This note is a Legal Tender at its Face Value in payment of all Debts, Public and Private, except when otherwise expressly stipulated in the Contract.

NOTE 4. As has been stated, the number of grains of pure silver in the silver dollar is $371\frac{1}{4}$. The silver half dollar, however, coutains only 173.01 grains, or nearly 7% less than one half the silver in the dollar. The quantities in the quarter dollar and the ten cent piece, moreover, are proportional to the quantity in the half dollar instead of to the quantity in the dollar.

The present ratio of the minor coins was established by an act of Congress passed Feb. 21, 1853. The purpose of the act was to check the exportation of silver, which was then taking place on account of the undervaluation of silver as compared with gold.

NOTE 5. Gold and silver in their pure state would be too soft for circulation as money. Each, therefore, is alloyed by a cheaper metal, 1 part of copper being added to 9 parts of silver to form silver coin, and 1 part of silver and copper to 9 parts of gold to form gold coin. The cent, moreover, is not a pure copper coin, but is composed of 95 parts copper and 5 parts tin and zinc.

NOTE 6. The division of the dollar into cents has already been explained. In referring to thousandths of a dollar the term MILL is occasionally employed, Thus, the tax voted by a town may be referred to as 9 mills on a dollar.

96. English, French, and German Money.

The unit of English money is the gold Sovereign, or Pound. This unit contains 4.8665 times as much gold as a United States dollar. The intrinsic value of an English pound is, therefore, $4.8665.

The unit of French money is the Franc, and of the German money the Reichmark. The value of the first unit is 19.3 cents, and of the second 23.8 cents. The money systems of both France and Germany like that of the United States, are decimal systems.

Table of English Money.

4 farthings (qr. or far.)	make	1 penny, *d.*
12 pence	"	1 shilling, *s.*
2 shillings	"	1 florin, *fl.*
5 shillings	"	1 crown, *cr.*
20 shillings	"	1 pound, £.
21 shillings	"	1 guiuea, *guin.*

The value of a pound being $4.8665, what is the value in dollars and cents

Of a guinea? Of a shilling

Of a crown? Of a florin? Of a penny? Of a farthing?

Table of French Money.

10 centimes	make	1 decime.
10 decimes	"	1 franc.

Table of German Money.

100 pfennige *(pennies)* make 1 reichmark *(mark.)*

Memorize the following approximate values of foreign coins.

A GUINEA is a little more and a SOVEREIGN a little less than $5.

A CROWN is considerably more than $1.

A FLORIN is a little less than 50 cents.

A SHILLING and a MARK are a little less and a FRANC is considerably less than 25 cents.

A PENNY is a little more and a DECIME a little less than 2 cents.

97. Measures of Quantity.

The standard unit of quantity should be some one of the standard units of volume. Not only is this not the case in the system of measures employed in England and the United States, but, to make the condition of affairs still worse, different standards are established according as the matter to be measured is in solid or in liquid form. The unit of dry measure is the **Bushel**, and of liquid measure the **Gallon.** The former unit is equivalent to a volume of 2150.42 cubic inches, and the latter to a volume of 231 cubic inches. The subdivisions of each of these units are shown in the following tables.

Measures of Capacity.

Liquid.			Dry.		
4 gills (gi.)	make	1 pint, pt.	2 pints	make	1 quart, qt.
2 pints	"	1 quart, qt.	8 quarts	"	1 peck, pk.
4 quarts	"	1 gallon, gal.	4 pecks	"	1 bushel, bu.

98. Relationship of Liquid, Dry, and Cubic Measures.

4 liquid quarts are how many cubic inches? 32 liquid quarts?

32 dry quarts are how many cubic inches?

What, then, is the ratio, to three decimal places, of 32 dry quarts to 32 liquid quarts?

What is the ratio of 1 dry quart to 1 liquid quart?

How many cubic inches are there in a cubic foot?

What, then, is the ratio, extended to thousandths, of a bushel to a cubic foot?

Increase 1728 cubic inches by $\frac{1}{4}$. What result do you obtain?

Increase 1848 cubic inches by $\frac{1}{6}$. What result do you obtain?

Memorize the following facts:

1. A gallon is 231 cubic inches.

2. A bushel is 2150.42 cubic inches, or nearly 1 1-4 cubic feet.

3. The ratio of a dry quart to a liquid quart is nearly 1 1-6.

* *

We wish to measure a quantity with two units of different magnitudes.

Suppose the larger unit to be twice the smaller. How will the number of the larger units in the quantity compare with the number of the smaller?

How will the number of the smaller units compare with the number of the larger if the ratio of the smaller unit to the larger

Is $\frac{3}{4}$? Is $\frac{7}{9}$? Is $\frac{5}{8}$? Is $\frac{4}{7}$?
Is $\frac{13}{23}$? Is $\frac{24}{37}$? Is $\frac{48}{66}$? Is $\frac{62}{79}$?

Give a general principle concerning the relation of the number of units in any quantity to the size of the units.

A box holds 3 bushels. About how many gallons will it hold? About how many cubic feet are there in the box?

A horse is fed 7 liquid quarts of grain a day. About how many days can it be fed from $1\frac{1}{4}$ bushels?

A keg holds 7 gallons. About what part of a bushel will it hold?

About how many dry quarts are equivalent to 14 liquid quarts?

About how many liquid quarts are equivalent to 18 dry quarts?

99. Measures of Weight.

Every particle of matter on or near the surface of the Earth is attracted toward the centre of the Earth by a constant force called gravity. The measure of this attraction is called **Weight.** Only one standard unit of weight should be used, and this standard should be the weight of some standard quantity of universally accessible matter. Neither of these

conditions is fulfilled in the system in use in the United States. As standards two independent units are employed, the Troy Pound and the Avoirdupois Pound, the value of the former being to the value of the latter as 5760 to 7000,

Table of Troy Weight.

24 grains (gr.)	**make**	**1 pennyweight, pwt.**
20 pennyweights	**"**	**1 ounce, oz.**
12 ounces	**"**	**1 pound, lb.**

Table of Avoirdupois Weight.

16 ounces (oz.)	**make**	**1 pound, lb.**
2000 pounds	**"**	**1 ton, T.**
2240 pounds	**"**	**1 long ton.**

NOTE 1. Dealers when purchasing coal obtain as a ton 2240 pounds, this quantity being called a LONG TON. In selling they commonly give only 2000 pounds as a ton, unless the purchaser insists upon 2240 pounds as the unit.

NOTE 2. Sugar is bought by retail dealers in barrels ranging from 212 to 250 pounds, and from 312 to 350 pounds. Molasses is bought by the barrel, tierce, and hogshead, the tierce holding about 75 gallons, and the hogshead from 125 to 150 gallons. Kerosene, which was formerly bought in barrels containing from 48 to 55 gallons, is now delivered by its principal producer, the Standard Oil Company, directly to dealers from portable tanks.

NOTE 3. A bushel of shelled corn weighs about 56 pounds. This number of pounds, therefore, is generally made the unit in purchasing corn from the producer. Meal, however, is sold by the hundred pounds, Oats are commonly sold by the bushel of 32 lb.

NOTE 4. The legal weight in Vermont of a bushel of wheat, potatoes, etc., is as follows:

Oats, 32 lb.; herdsgrass or timothy seed, 45 lb.; apples, 46 lb.; barley and buckwheat, 48 lb.; carrots, 50 lb.; onions, 52 lb.; rye or Indian corn 56 lb.; wheat, potatoes, beets, turnips, peas, clover seed, 60 lb.; beans, 62 lb.

The corresponding list in New Hampshire is as follows:

Oats, 32 lb.; corn and rye meal, 50 lb.; corn and rye, 56 lb.; wheat, potatoes, peas, 60 lb.

NOTE 5. In avoirdupois weight, the term DRAM is occasionally applied to 1-16 of an ounce. The finest avoirdupois scales, however, are not commonly furnished with weights of less than ⅛ ounce.

100. Relationship of Troy, Avoirdupois, and Cubic Measures.

A cubic inch of water at 62 degrees weighs 252.458 grains.

By what must we multiply 252.458 to obtain the weight in grains of a cubic foot of water?

By what must we divide our product to find the weight in troy pounds of a cubic foot of water?

Cancel the common factors in the divisor and the dividend, and complete the indicated operations. What do you find to be the weight of a cubic foot of water in troy pounds?

What is the ratio, in its simplest form, of an avoirdupois pound to a troy pound?

What, then, will be the ratio of the weight of a cubic inch of water in avoirdupois pounds to its weight in troy pounds?

$252.458 \times 12 \times 12 \times 12 \times \frac{1}{24} \times \frac{1}{20} \times \frac{1}{12} \times \frac{144}{175} = ?$

What, then, is the weight of a cubic foot of water in avoirdupois pounds?

Memorize the following facts:

1. A cubic foot of water weighs about 62 1-3 avoirdupois pounds.

2. The ratio of a troy to an avoirdupois pound is 144-175.

101. Apothecaries' Weight.

Apothecaries commonly purchase drugs by avoirdupois weight but compound them by troy weight. When thus used the troy ounce is divided into 8 parts called DRAMS, and the dram into 3 parts called SCRUPLES.

The number of grains in a troy ounce is the product of what two factors?

The number of scruples in an ounce equals which of these factors?

The number of grains in a scruple, then, must equal which factor?

How many grains, then, are there in a scruple?

Give the table of troy weight as used by apothecaries.

102. Apothecaries' Measure.

A gallon of water at the temperature of its greatest density weighs 8.3389 avoirdupois pounds, thus making the weight of a pint a little more than a pound. By reason of this coincidence, the custom has arisen of dividing a pint into 16 parts and calling one of these parts a FLUID-OUNCE. The fluid-ounce is divided into 8 parts called FLUID-DRAMS, thus being

treated as a troy instead of an avoirdupois unit—a treatment in line with the lack of intelligent method and design on which our entire system of weights and measures seems to be based.

The fluid dram is divided into parts called MINIMS.
How many minims must there be in a dram of capacity that the minim may be the same part of the ounce of capacity that the grain is of the troy ounce of weight?
Write out a table of Apothecaries' Fluid Measure.

NOTE. A fluid-dram is about a teaspoonful, and a minim about a drop.

Find by testing the number of drops in a teaspoonful.

Ex. 118.

1. Add 37 bu. 3 pk. 1 pt., 18 bu. 6 qt. 1 pt., 23 bu. 2 pk. 4 qt., 86 bu. 2 pk. 7 qt. 1 pt.

2. Add 35 gal. 3 qt. 2 pt. 3 gi., 16 gal. 2 qt. 1 pt. 3 gi., 19 gal. 1 qt. 2 gi., 27 gal. 1 gi.

3. Add 37 lb. 7 oz. 16 pwt. 16 gr., 19 lb. 11 oz. 17 pwt. 21 gr., 11 lb. 5 oz. 23 gr., 32 lb. 2 oz. 13 pwt. 17 gr.

4. Subtract 7 T. 356 lb. from 12 T. 214 lb.

5. Multiply 15 bu. 2 pk. 5 qt. 1 pt. by 17.

6. Reduce $\frac{5}{8}$ of a gallon to integers of lower denominations.

7. Divide 71 bu. 3 pk. 4 qt. 2 pt. by 27.

8. Divide 48 gal. 5 qt. 1 pt. 2 gi. by 15.

9. Reduce 10 oz. 17 pwt. 13 gr. to a fraction of a pound.

10. Reduce .137 of a bushel to integers of lower denominations.

11. Reduce 2324 gills to units of higher denominations.

12. Reduce 7236 grains to integers of higher denominations.

13. Reduce 736 dry pints to integers of higher denominations.

14. Subtract £25 17s. 11d. 3 far. from £57 6s. 8d. 2 far.

15. Reduce .59 of a pound to integers of lower denominations.

16. Change 43861 grains to avoirdupois pounds.

16. A mass of gold weighs 1 lb. 5 oz. 15 pwt. 16 gr. What is its weight in avoirdupois pounds and ounces?

103. Measures of Time.

Watch the course of the sun through a day What will you observe as to its position with reference to the horizon at sunrise? at noon? at any intermediate instant?

What will you observe as to its relative positions at noon, at sunset, and at any intermediate instant?

The apparent daily motion of the sun about the earth is caused by the daily rotation of the earth upon its axis. This motion is uniform, and its direction is oppoaite to the direction of the hands of a watch whose face is held toward the northern horizon, or from right to left to the observer facing the south.

Suppose the only motion of the earth to be its rotation about its axis. How would the intervals between any two consecutive instants at which the sun is seen at its greatest height compare with each other?

Instead, the earth has also a motion about the sun; this motion being from left to right to the observer facing the south at noon. We wish to find the effect, if any, of this motion upon the length of our days.

* *

Stick a pin into an apple half way between its two ends. Place the apple so that the pin will point at some fixed object from three to five feet distant.

Standing behind the apple, turn it on its base through a complete rotation in such a direction that the half toward the fixed object will turn from right to left. While so turning it, move it about the object from left to right through $\frac{1}{4}$ of a revolution.

At the end of the rotation will the pin again point at the fixed object?

Assuming that the apple does not continue its revolution, through what part of an additional rotation must it be turned that it may again point at the object?

How would the amount of additional rotation have compared with this amount

Had the apple been moved less than a fourth of a revolution?

Had it been moved more than half of a revolution?

* *

Suppose the earth while rotating once on its axis to move about the sun through $\frac{1}{365}$ of a revolution. Will the sun occupy the same position relative to the earth at the end of the rotation that it occupied at the beginning?

Through what part of an additional rotation must the earth turn that the relative position may become the same?

How, then, does the length of the solar day compare with the time required for the earth to turn upon its axis?

The path of the earth about the sun is not exactly a circle. In consequence, its rate is not uniform, but varies with its solar distance. How, then, will the amounts of additional rotation compare with each other?

How will the intervals of time required for the complete rotations plus the additional rotations compare with each other?

A comprehension of the preceding facts and principles will make clear the basis of the unit of time. This unit is the **Mean Solar Day**, or the average interval between consecutive instants at which the sun reaches its greatest daily altitude above the southern horizon.

* * *

A second unit of time, called the **Tropical Year**, or the interval between two consecutive instants at which the axis of the earth has the same inclination to the sun's rays. This interval nearly coincides with the time in which the earth makes a revolution about the sun, but slightly differs from it on account of a slight regular variation in the direction of the earth's axis.

The length of the tropical year, to the nearest second, is 365 days, 5 hours, 48 minutes, and 50 seconds. As this compound number would be an inconvenient unit, the **Civil Year**, composed of that number of complete days nearest the length of the tropical year is substituted for it. The laws governing the length of this year are developed in the following inductive exercises:

What number of complete days is nearest the length of the tropical year?

This number of days is the ordinary civil year. How much less is it than the tropical year?

By what integer must 5 h. 48 min. 50 sec. be multiplied to give as a product nearly a complete day?

Suppose an extra day to be added to a fourth year. That day will be how many minutes and seconds more than 4 times the extra 5 hr. 48 min. 50 sec. of the tropical year?

44 min. 40 sec. are what fraction of an hour?

A civil year of 366 days is called a **Leap Year.** By what fraction of an hour, then, do 3 common civil years and 1 leap year exceed 4 tropical years?

We have found the error in a system of years composed of 3 common years and 1 leap year to be $\frac{67}{90}$ of an hour for each 4 years. We wish to find the error for 400 years.

What is the ratio of 400 to 4?

100 times $\frac{67}{90}$ of an hour equals how many days?

Suppose that we make 3 of the 4 years ending in two 0's common years instead of leap years. What will then be the error in our system in 400 years?

$\frac{11}{108}$ is a little more than how many tenths?

If, then, there is an error in our system after its second modification of about $\frac{1}{10}$ day in 400 years, in about how many times 400 years will there be an error of an entire day?

10 times 400 equals what?

In about how many years, then, from its beginning should a third correction be made in our system of time?

How should this correction be made?

What would be the exact error after this correction?

The instant at which the earth's axis is most inclined to the sun's rays, and at which the sun reaches its greatest annual height above the southern horizon, is called the **Summer Solstice,** and the corresponding instant of least inclination of the axis is called the Winter Solstice. The summer solstice of 1901 was, in Eastern Time, on June 21, at 2 hours and 23 minutes after midnight, and the winter solstice at 36 minutes past 7 in the forenoon of Dec. 22.

We have found that under the present system we reckon

(1) For an ordinary year, 365 days.

(2) For every year divisible by 4 and not ending in two 0's, 366 days.

(3) For every year ending in two 0's and divisable by 400, 366 days.

We have also found that by so reckoning we have made the civil year too long by about $\frac{1}{4000}$ day. When, then, will June 21 arrive with reference to the summer solstice in about 4000 years? When, if no change should be made in the system, in 30 times 4000 years? In how many times 4000 years would June 21 arrive at the winter instead of at the summer solstice?

NOTE 1. The present calendar was invented by Julius Caesar, and revised by Pope Gregory, who also added 10 days to the date under the old system. This revision was not accepted by the English Parliament until 1752, when it was necessary to add 11 days instead of 10.

NOTE 2. The civil year is divided into the following 12 MONTHS.

January,	31 da.	May,	31 da.	September,	30 da.
Febuary, 28 or	29 "	June,	30 "	October,	31 "
March,	31 "	July,	31 "	November,	30 "
April,	30 "	August,	51 "	December,	31 "

Observe that April, June, September, and November have each 30 days, and that each of the remaining months except February has 31 days.

Table of Time Measure.

60 seconds (sec.)	make	1 minute, min.
60 minutes	"	1 hour, h.
24 hours	"	1 day, d.
7 days	"	1 week, wk.
375 days	"	1 common year, yr.
366 days	"	1 leap year, l. y.
100 years	"	1 century.

104. To Find the Time in Years, Months, and Days between two Dates.

We wish to find the time in years, months, and days from June 13, 1887 to Dec. 28, 1895.

What is the time from June 13, 1887 to June 13, 1895?

From June 13, 1895 to Dec. 13, 1895?

From Dec. 13, 1895 to Dec. 29, 1895?

What, then, is the time from June 13, 1887 to Dec. 29, 1895?

NOTE. To determine the number of months, mentally name each after the first to and including the last, and in doing this mentally accent each second month. Thus, in the preceding problem, July *August;* September, *October;* November, *December.*

Using the pencil or crayon only to record in order the number of years, months, and days, give the time

From May 24, 1892 to Aug. 29, 1897.

From Aug. 7, 1896 to Dec. 23, 1899.

From Jan. 4, 1893 to Mar. 10, 1895.

From Apr. 7, 1881 to June 24, 1893.

From Sept. 24, 1889 to Dec. 28, 1896.

From Feb. 4, 1895 to Sept. 16, 1998.

* *

We wish to find the time from Jan. 23, 1893 to Oct. 23, 1893.

From Jan. 23, 1893 to Jan. 23, 1894 is how many months?

From Oct. 23, 1893 to Jan. 23, 1894?

From Jan. 23, 1893 to Oct. 23, 1893 is, then, 12 months less how many months?

12 less 3 equals what?

From Jan. 23, 1893 to Oct. 23, 1893 is, then, how many months?

What, evidently, is the most convenient method of finding the number of months when greater than 6?

Using the pencil or crayon as before directed give the time
From Feb. 6, 1895 to Dec. 10, 1899.
From March 15, 1893 to Nov. 26, 1897.
From Jan. 23, 1896 to Oct. 30, 1899.
From Apr. 16, 1894 to Dec. 28, 1898.
From Jan. 15, 1890 to Dec. 21, 1894.

* *

We wish to find the time from July 23, 1890 to May 27, 1899.

Suppose that in determining the number of years we reckon to July 23, 1899. Will this be correct?

Give the reason for your answer.

To what date, then, shall we first reckon?

Complete the problem, explaining each step as you perform it.

Using the pencil or crayon only as before directed give the time
From Nov. 21, 1902 to March 27, 1905.
From June 3, 1903 to May 24, 1907.
From Aug. 20, 1904 to June 25, 1906.
From Dec. 4, 1901 to Apr. 22, 1904.

We wish to find the time from June 24, 1902 to Oct. 20, 1902.

Suppose that in determining the number of months we reckon to Oct. 24. Will this be correct?

Give the reason for your answer.

To what date, then, shall we first reckon?

Complete and explain the problem.

Using the pencil or crayon only as before directed give the time
From Oct. 27, 1896 to Nov. 19, 1899.
From Oct. 24, 1887 to Dec. 21, 1897.
From Aug. 16, 1892 to Apr. 15, 1899.
From May 16, 1895 to July 1, 1897.

NOTE. In finding the time between two dates in years, months, and days, each month, whatever its length, is considered to be 30 days. Thus, in the first of the preceding problems, in finding the number of days we reckon from Oct. 27 to Oct. 30, instead of to Oct. 31, and combine the 3 days thus obtained with the 10 days in November.

If the days are more than 15 they may be conveniently found by thinking of them as a full month and making the necessary correction. Thus, in the second problem, the number of days from Oct. 24 to Dec. 21 is 30 less 3, or 27.

When the days are a little more or a little less than 15 it is well to employ both processes and compare the results. Thus, in the fifth problem the days are 12 plus 4 or 30 less 14, according to the

method employed. The result by each method being 6 days, there can be but little doubt of its correctness.

	SOLUTIONS.		
Ex. 1. Find the time from	(1)	(2)	(3)
June 28, 1893 to Sept. 8, 1899.	6—2—10	4—8—25	2—11—26

Ex. 2. Find the time from Nov. 10, 1891 to Aug. 5, 1896.

Ex. 3. Find the time from Feb. 17, 1892 to Feb. 13, 1895.

EXPLANATIONS.

Ex. 1. From 1893 to 1899 is 6 years; from June to August is 2 months; and from the 28th of a month to the 8th of the following month is 2 plus 8, or 10, days.

Ex, 2. From 1891 to 1895 is 4 years; from November to July is 12 less 4, or 8 months; and from the 10th of a month to the 5th of the following month is 30 less 5, or 25, days.

Ex. 3. From Feb. 17, 1892 to Feb. 13, 1895 is 3 years less 4 days, or 2 yr. 11 mo. 26 da.

Chronological Table of Important Events in American History.

Give the time in years, months, and days between each two consecutive dates contained in this table.

1492 Oct. 12—America discovered, at San Salvador, by Columbus.
1512 March 27—Florida discovered and named by Ponce de Leon.
1565 Aug. 29—First permanent settlement made at St. Augustine.
1607 May 13—Jamestown settled by English under London Co.
1619 July 30—First representative assembly at Jamestown.
1620 Dec. 21—Landing of the Pilgrims.
1638 Apr. 18—New Haven settled; Sept. 14, Harvard founded.
1643 May 19—United Colonies of New England formed.
1681 March 4—Pennsylvania granted to William Penn.
1684 June 18—Massachusetts' charter recinded.
1704 Apr. 24—First permanent American newspaper established.
1755 July 9—Braddock's defeat in expedition against Ft. Duquesne.
1759 Sept. 18—Quebec surrendered to English under Wolfe.
1765 March 22—Stamp Act passed by Parliament.
1770 March 5—The Boston Massacre.
1774 Sept. 5—First Colonial Congress.
1775 Apr. 19—Battle of Lexington.
1776 March 18—Boston Evacuated by the British.

1776 July 4—Declaration of Independence passed.
Sept. 15—Evacuation of New York bv the Americans.
Dec. 26—Battle of Trenton won by Washington.
1777 Oct. 17—Surrender of Burgoyne to Americans under Gates.
1778 Feb. 7—Treaty of Alliance with France signed.
June 18—Philadelphia evacuated by British.
1781 Oct. 19—Surrender of Cornwallis at Yorktown.
1783 Sept. 3—Treaty of Peace with England signed at Paris.
1789 Apr. 30—Washington inaugurated first President.
1791 March 4—First State (Vermont) admitted to Union.
1799 Dec. 14—Death of Washington at Mount Vernon.
1803 Apr. 30—Louisiana purchased by the U. S.
1812 June 19—War with Great Britain declared.
1814 Aug. 24—Washington, D. C., captured by the British.
Dec. 24—Treaty of peace signed at Ghent, Belgium.
1815 Jan. 8—Battle of New Orleans won by Jackson.
1820 March 3—Missouri Compromise passed.
1826 July 4—Death of Adams and Jefferson.
1846 Apr. 26—Mexican War begun.
July 4—California declared independent.
1848 Feb. 2—Treaty of Peace with Mexico signed.
1850 Sept. 9—Bill repealing Missouri Compromise passed.
1858 Aug. 16—First message sent by the Atlantic Cable.
1860 Dec. 20—Secession voted by South Carolina.
1861 Apr. 12—Fort Sumpter fired upon by Confederates.
July 21—Defeat of Federals at Bull Run.
1862 June 26 to July 1—Seven days' battles before Richmond.
1863 Jan. 1—Emancipation Proclamation by President Lincoln.
1863 July 1-2—Defeat of Lee at Gettysburg.
July 4—Surrender of Vicksburg to Grant.
1865 Apr. 9—Surrender of Lee to Grant at Appomatox.
1872 Sept. 14—Geneva Award against England of $15,500,000.
1876 May 10—Opening of Centennial Exposition at Philadelphia.
1879 Jan. 1—Resumption of specie payments.
1883 Oct. 1—Two Cent Letter Postage established.
1893 May 1.—Opening of Columbian Exposition at Chicago.
1898 Feb. 15—Destruction of battleship Maine at Havana.
1898 Apr. 19—Declaration of war against Spain.
1898 Apr. 30—Phenomenal victory of Dewey at Manila.
1898 July 4—Destruction of Spanish fleet at Santiago.
Aug. 12—Signing of Peace Protocol.

105. Measures of Angles.

The unit in measurement of angles is that angle which is produced by $\frac{1}{360}$ of a revolution, and which is measured by $\frac{1}{360}$ of a circumference.

Table of Angular Measure.

60 seconds (″)	make	1 minute, ′.
60 minutes	”	1 degree, °.
360 degrees	”	1 Circumference.

106. Miscellaneous Measures.

12 units	make	1 dozen.	24 sheets	make	1 quire.
12 dozen	”	1 gross.	20 quires	”	1 ream.
12 gross	”	1 great gross.	5 reams	”	1 bundle.
20 units	”	1 score.	2 bundles	”	1 bale.

Complete and read the following tables:

* * in.	make	1 ft.	* * lb.	make	1 l. T.	* * far.	make	1 *d.*
* * gi.	”	1 pt.	* * pwt.	”	1 oz.	* * rd.	”	1 mi.
* * yd.	”	1 rd.	* * gr.	”	1 pwt.	* * rd.	”	1 ch.
* * ft.	”	1 yd.	* * in.	”	1 hand.	* * sec.	”	1 min.
* * °.	”	1 cir.	* * pt.	”	1 qt.	* * ″.	”	1 ′.
* * min.	”	1 h.	* * lb.	”	1 T.	* * ′.	”	1°.
* * rd.	”	1 fur.	* * qt.	”	1 gal.	* * da.	”	1 c. yr.
* * *s.*	”	1 £.	* * fur.	”	1 mi.	* * pk.	”	1 bu.

* * sq. ft.	make	1 sq. rd.	* * oz.	make	1 tr. lb.
* * sq. ch.	”	1 A.	* * sq. ft.	”	1 sq. yd.
* * cu. in.	”	1 cu. ft.	* * av. oz.	”	1 lb.
* * cu. ft.	”	1 cu. yd.	* * cu. ft.	”	1 cd.
* * sq. yd.	”	1 sq. rd.	* * sq. rd.	”	1 A.

An eagle contains * * grains of pure gold.
A silver dollar contains * * grains of pure silver.
The value of an English £ is * * dollars.
The value of a franc is * * cents.
The value of a mark is * * cents.
A gallon is * * cubic inches, and a bushel * * cubic inches.
The ratio of a troy to an avoirdupois pound is * * .
The ratio of a circumference to its diameter is * * .
A cone or pyramid is * * the corresponding cylinder or prism.
The area of a circle is the product of its * * and its * * .
A scruple is * * grains, or * * of a dram, or * * of an ounce.
A fluid-ounce is * * of a pint.
A fluid-dram is * * minims, or * * of a fluid-ounce.

107. Effects of the Rotation of the Earth on Its Axis.

We have spoken of the instant at which the earth's axis is most inclined toward the sun's rays, or the summer solstice; and of the instant at which it is the least inclined, or the winter solstice. Two other instants are of equal importance. These are the two instants in each year at which the line connecting the centre of the sun and the centre of the earth is at right angles to the earth's axis, and are called, as they occur in the spring or in the autumn, the **Vernal** or the **Autumnal Equinox.**

Diagram 4, on page 248, represents the earth at either equinox. The points *N* and *S* in the diagram are the two ends, or **Poles,** of the earth's **Axis,** the point *N* being the end which points to a certain bright star called the **North Star.** The line in the diagram composed of the points on the earth's surface which are half way between the two poles is called the **Equator.**

Suppose the earth to make a complete rotation at either equinox. During what part of the rotation will every point on the surface of the earth be exposed to the sun's rays?

What, then, will be the length of the day and of the night at every point on the earth's surface?

The vernal equinox of 1901 was on March 21, at 2 h. 23 m. A. M.; and the autumnal equinox on Sept. 23, at 1 h. 9 m. P. M. What, nearly, was the length of the day and of the night on each of these dates?

At nearly what hour did the sun rise and set on each date?

* *

The direction the earth is turning on its axis, or from which the sun appears to be moving in its course, is called **East;** and the direction from which the earth is turning, or in which the sun appears to be moving, is called **West.** Think of the figure representing the earth in Diagrams 1 and 2 as turning in a direction opposite to the direction in which the hands of a watch would turn if held between yourself and the diagram.

Examine the diagrams and give the direction
- Of Newfoundland from Vermont.
- Of Newfoundland from Italy.
- Of Italy from China.
- Of China from Vermont.
- Of Alaska from Vermont.
- Of California from Chicago.

Each diagram represents the earth at one of the equinoxes. At what time, then, does the sun rise at each locality represented in the diagrams?

1. Diagram Showing Time in Other Localities at Sunrise in New England.

When it is sunrise in Vermont what time is it
In Newfoundland? In Italy?

When it is noon at Chicago what time is it
In Vermont? In Alaska? In California?
In Newfoundland?

When it is midnight in Vermont what time is it
In Italy? In Alaska? In Chicago?
In Newfoundland? In California?

2. Diagram Showing Time in Other Localities at 1 P. M. in New England.

When it is a certain hour in any locality what will be the relative time at any locality to the east? to the west?

Give a general principle concerning the relative solar times of two localities.

* *

Lines drawn on the surface of the earth directly from pole to pole are called **Meridians**, this name being given from the fact that all points in the same meridian have their mid-day, or noon, at the same instant.

Lines drawn on the surface of the earth parallel to the equator are called **Parallels of Latitude**. The angular distance of a locality from the equator is called its **Latitude**, and its angular distance from a standard meridian its **Longitude**. The standard meridian in most general use is that passing through the Astronomical Observatory at Greenwich, near London.

Latitude is reckoned from the equator to the poles, and longitude from a standard, or **Prime**, meridian, to the meridian half way around the earth. What, then, is the greatest latitude a place can have? the greatest longitude?

How, evidently, shall we find the difference in the latitude of two places that are on the same side of the equator, or the longitude of two places that are in the same direction from the prime meridian?

One of two localities is a given number of degrees north of the equator and the other a given number of degrees south of the equator. How shall we find their difference in latitude?

One of two localities is in 85 degrees east longitude and the other in 89 degrees west longitude. How shall we find their difference in longitude?

One locality is in 45° east longitude, and the other in 145° west longitude. We wish to find their difference in longitude.

Suppose that we add the two longitudes. What sum shall we obtain?

How many degrees are there in a half circumference? in an entire circumference?

If, then, it is 185 degrees from the first locality to the second in one direction, how many degrees must it be in the other direction?

One locality is in 170 degrees east longitude, and another in 160 degrees west longitude. We wish to find their difference in longitude.

How far is the first locality from the meridian of 180 degrees?

How far is the second locality from this meridian?

How many degrees apart, then, are the two localities?

In what two ways may we find the difference in longitude of two places on opposite sides of the prime meridian when the sum of their longitudes is more than 180 degrees?

* *

Through how many degrees does the earth turn in a day?
How many hours are there in a day?
Through how many degrees, then, does the earth turn in an hour?
What, then, is the difference in the longitude

Of Vt. and Newfoundland?	Of Vt. and California?
Of Vt. and Italy?	Of Vt. and Alaska?
Of Italy and China?	Of Chicago and Italy?
Of Vt. and China?	Of Cal. and Italy?
Of Cal. and China?	Of Alaska and Italy?

The longitude of Chicago is about 88° west. What, then, is the longitude

Of California?	Of China?
Of Alaska?	Of Newfoundland?
Of Italy?	Of Vermont?

NOTE. Find on a map the longitude of the preceding localities. Observe that the longitude you obtain for Alaska is that of the western part of the Aleutian Islands.

108. To Find Difference in Longitude when Difference in Time is Given.

A difference of an hour in the solar time of two localities is produced by a difference of how many degrees of longitude?
How, then, shall we change difference in time expressed in hours to difference in longitude expressed in degrees?

A difference of a sixtieth of an hour in time is produced by a difference of how many sixtieths of a degree of longitude?
A difference of a minute of time, then, is produced by a difference of how many minutes of longitude?
How, then, shall we change difference in time expressed in minutes of time to difference in longitude expressed in minutes of longitude?

A difference of a sixtieth of a minute of time is produced by a difference of how many sixtieths of a minute of longitude?
A difference of a second of time, then, is produced by a difference of how many seconds of longitude?
How, then, shall we change difference in time expressed in seconds of time to difference in longitude expressed in seconds of longitude?

* *

We wish to find what difference in longitude will produce a difference of a given number of hours, minutes, and seconds in time.
By what must we multiply the given number of seconds of

time to obtain the number of seconds in the required difference in longitude?

By what must we divide our product to reduce it to the next higher denomination?

Multiplying by 15 and dividing by 60 is equivalent to dividing by what number?

How, then, might we at a single step have changed the given number of seconds of time to minutes of longitude?

How, following the same method, may we at one step change the given number of minutes of time to degrees of longitude?

Suppose our minutes of time to be 39 and our seconds of time to be 46. What is the largest multiple of 4 less than 39? less than 46?

How many of our seconds of time, then, can we change to an integral number of minutes of longitude? how many of our minutes of time to an integral number of degrees of longitude?

What shall we do with the remaining seconds of time? with the remaining minutes of time?

Give a rule for finding the difference in longitude expressed in degrees, minutes, and seconds that will produce a given difference in time expressed in hours, minutes and seconds.

When it is noon in New York, it is 6 h. 24 m. 36 sec. in Honolulu. The longitude of New York is 74° 3′ W. What is the longitude of Honolulu?

SOLUTION.

8°	6′	
5 h.	35 m.	24 sec.
83°	51′	
157°	54′	

EXPLANATION.

The difference in the time of New York and of Honolulu is 12 hours less 6 h. 24 m. 36 sec., or 5 h. 35 m. 24 sec.

The difference of an hour in time is produced by a difference of 15 degrees in longitude, and the difference of any part of an hour is produced by 15 times that part of a degree. Therefore, to find the difference in the longitude of the two localities expressed in degrees, minutes, and seconds we multiply by 15 their difference in time expressed in hours, minutes, and seconds.

A difference of 24 seconds of time is produced by a difference of 15 times 24 seconds of longitude, or of $\frac{1}{60}$ of 15 times 24, or $\frac{1}{4}$ of 24, or 6, minutes of longitude.

A difference of 35, or 32 plus 3, minutes of time is produced by a difference of 15 times (32 plus 3) minutes of longitude.

15 times 3 minutes of longitude, plus the 6 minutes of longitude previously obtained, are 51 minutes of longitude.

15 times 32 minutes of longitude equal $\frac{1}{60}$ of 15 times 32, or $\frac{1}{4}$ of 32, or 8, degrees of longitude.

A difference of 5 hours of time is produced by a difference of 15 times 5, or 75, degrees of longitude. These combined with the 8 degrees previously obtained give 83 degrees. The difference in the longitude of the two places, therefore, is 83° 51′.

The sun, evidently, on the given day rose in New York before it rose in Honolulu. New York, therefore, as compared with Honolulu, is in the direction from which the sun appears to rise, or in an easterly direction, and Honolulu is west of New York. The longitude of Honolulu, therefore, is 74° 3′ plus 83° 51′, or 157° 54′, W.

NOTE. Each new civil day begins on the west side of the 180th meridian from Greenwich, and ends, 48 hours later, on the east side of that meridian. For example, a Sunday commencing at a certain instant at the Antipodes Island, south-east of New Zealand, will begin 6 hours later in the Bay of Bengal, 12 hours later at London, 18 hours later at St. Louis and Memphis, and nearly 24 hours later in the central groups of the Aleutian Islands.

Reflection upon the following questions will help to make clear the peculiar nature of the civil day.

Two boys on opposite sides of the 180th meridian are playing ball. The boy on the west side throws the ball at 2:30 P.M., Tuesday. When is it caught by the boy on the east side?

They play again, and the ball is thrown at 2:30 P. M., Friday by the boy on the east side. When is it caught by the boy on the west side?

Suppose a man to be able to travel around the earth in 24 hours. He starts on Friday morning from the west side of the 180th meri-and travels to the west. When will he arrive at the east side of the meridian?

Suppose that he starts on Friday morning from the east side, and travels to the east. When will he arrive at the west side?

How many times will he see the sun rise and set in travelling westward around the earth? in travelling eastward?

Two persons start from the same point and travel at the same rate, one toward the east and the other toward the west. Will they have travelled the same distauce at sunset? If not, which will have travelled the farther?

Suppose that they travel until they have travelled around the earth. Where will the one travelling to the west lose the time that he will seem to have been gaining? Where will the one travelling to the east gain the time that he will seem to have been losing?

Ex. 119.

The accompanying table gives the relative time of 10 localities. The longitude of the first, Boston, is 71° 3′ 30′′ west. Find the longitude of each of the other nine.

Boston	**12 m.**
Albany	**11 h. 49 m. 15 sec. a. m.**
Ann Arbor	**11 h. 9 m. 22 sec. a. m.**
Berlin	**5 h. 37 m. 49 sec. p. m.**
Cincinnati	**11 h. 6 m. 16 sec. a. m.**
Chicago	**10 h. 53 m. 43 sec. a. m.**
Paris	**4 h. 53 m. 34 sce. p. m.**
Washington	**11 h. 46 m. 13 sec. a. m.**
Havana	**11 h. 14 m. 48 sec. a. m.**
Guam	**26 h. 22 m. 48 sec. a. m.**

109. To Change Difference in Longitude to Difference in Time.

How many degrees of longitude does it take to produce a difference of an hour in time?

How, then, shall we change a difference in longitude expressed in degrees to difference in time expressed in hours?

How many sixtieths of a degree of lcngitude will be required to produce a difference 1 sixtieth of an hour in time?

How many minutes of longitude, then, will be required to produce a difference of 1 minute of time?

How, then, shall we change a difference in longitude expressed in minutes to a difference in time expressed in minutes?

How many sixtieths of a minute of longitude will be required ed to produce a difference of 1 sixtieth of a minute in time?

How many seconds of longitude, then, will be required to produce a difference of 1 second in time?

How, then, shall we change a difference in longitude expressed in seconds to a difference in time expressed in seconds?

* *

We wish to find what difference in time will be produced by a difference of a given number of degrees, minutes, and seconds in longitude.

By what must we divide the difference in degrees to obtain the number of hours in the required difference in time?

Suppose that we wish to change our difference in degrees to difference in time expressed in minutes. By what shall we multiply the quotient obtained in dividing the given number of degrees by 15?

Dividing by 15 and multiplying by 60 is equivalent to multiplying by what number?

How, then, can we at a single step change a difference in longitude expressed in degrees to a difference in time expressed in minutes?

Explain according to the same principle the change of a difference in longitude expressed in minutes to a difference in time expressed in seconds.

Suppose our degrees to be 37, and our minutes of longitude to be 49. How many of our degrees can we change to an integral number of hours? how many of our minutes of longitude to an integral number of minutes of time?

What shall we do with the remaining degrees? with the remaining minutes of longitude?

Give a rule for finding the difference in time expressed in hours, minutes, and seconds that will be produced by a given difference in longitude expressed in degrees, minutes, and seconds.

The longitude of Manila is 120 degrees 58 minutes 3 seconds east, and of Boston 71 degrees 3 minutes 30 seconds west. The naval battle won by Admiral Dewey at Manila was fought on Apr. 30, 1898, and was opened at 5 a. m. If intelligence of the opening of the battle could have been conveyed instantaneously to Boston when would it have been received there?

SOLUTION.

4 sec.
192° 1′ 33″
12 h. 48 m. $6\frac{1}{5}$ sec.
29 d. 4 h. 11 m. $53\frac{4}{5}$ sec. a. m.

EXPLANATION.

The difference in the longitude of Manila and of Boston is 120 degrees 58 minutes 3 seconds plus 71 degrees 3 minutes 30 seconds, or 192 degrees 1 minute 33 seconds.

A difference of 15 degrees in longitude is required to produce a difference of an hour in time, and to produce a difference of any part of an hour a difference of 15 times that part of a degree is required.

A difference of 192, or 180 plus 12, degrees will produce a differ-

ence of $\frac{1}{15}$ of (180 plus 12) hours; and a difference of 180 degrees will produce a difference of $\frac{1}{15}$ of 180, or 12, hours.

A difference of 12 degrees will produce a difference of 60 times $\frac{1}{15}$ of 12, or 4 times 12, or 48, minutes of time.

A difference of 1 minute of longitude will produce a difference of 60 times $\frac{1}{15}$ of 1, or 4 times 1, or 4, seconds of time; and a difference of 33 seconds of longitnde will produce a difference of $\frac{1}{15}$ of 33, or $2\frac{1}{5}$ seconds of time. The number of seconds in the required difference in time, therefore, is 4 plus $2\frac{1}{5}$, or $6\frac{1}{5}$, and the required difference in time is 12 h. 48 m. $6\frac{1}{5}$ sec.

Manila is in the east, or in the direction of the rising sun, from Boston. At Boston, therefore, the instant of beginning the engagement lacked 12 h. 48 m. $6\frac{1}{5}$ sec. of being 5 a. m. of Apr. 30, and was at 4 h. 11 m. $53\frac{4}{5}$ sec. on the afternoon of Apr. 29.

Ex. 120.

1. Find what time it is in each of the remaining cities in the list when it is noon in New York.

2. Find what time it is in each of the cities when it is noon at your home.

Mexico City	99	degrees	5	minutes	00	seconds	West
New York City	74	"	3	"	00	"	
New Orleans	90	"	2	"	30	"	
Rome	12	"	27	"	00	"	East
Richmond	77	"	25	"	45	"	West
San Francisco	122	"	26	"	45	"	
St. Paul	95	"	4	"	55	"	
St. Louis	90	"	15	"	15	"	
Calcutta	88	"	19	"	2	"	East
Jerusalem	35	"	32	"	00	"	

110. Standard Time.

Name some of the States and some of the large cities within $7\frac{1}{2}$ degrees of the 75th meridian; of the 90th meridian; of the 105th meridian; of the 120th meridian.

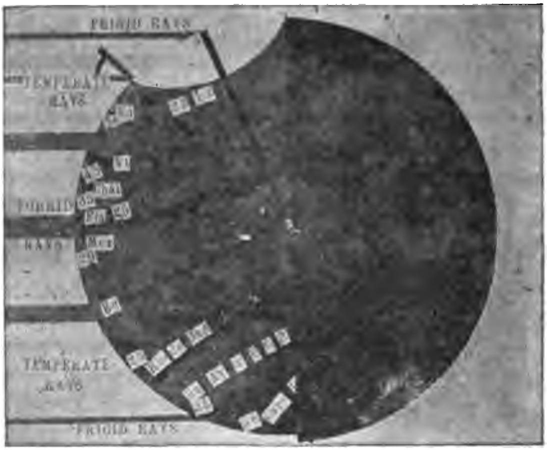
As we have learned, only localities in the same longitude have the same solar time. To avoid the confusion arising from this fact a system of **Standard Time** has been established in the United States. Under this system a locality is supposed to be in **Eastern Time, Central Time, Mountain Time,** or **Pacific Time,** according as it is within $7\frac{1}{2}$ degrees of the 75th, the 90th, the 105th, or the 120th meridian.

Diagrams Illustrating Latitude, Length of Day, and Changes of Seasons.

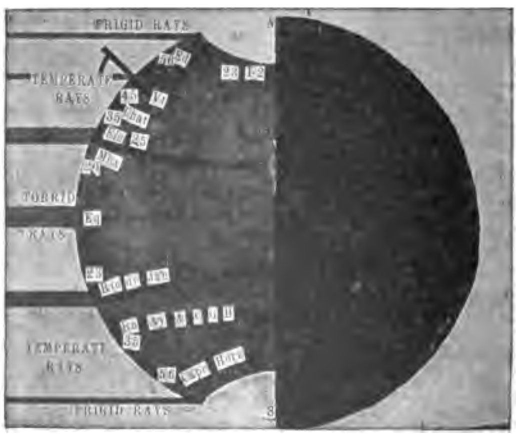
Note. Of the terms *axis, equator, and poles*, which occur in the accompanying diagrams, explanation has been given.

By *torrid, frigid*, and *temperate* rays, is signified those rays of the sun which at the equinoxes fall respectively upon the torrid, frigid, and temperate zones.

Use these diagrams to assist you in developing the laws concerning latitude, length of days, changes of seasons, etc., treated in the following pages.



3. Earth at Summer Solstice.



4. Earth at Equinoxes.

111. To Determine Latitude.

Suppose an observer to be at the North Pole. Where will he see the North Star?

Suppose him to travel 1 degree toward the equator. The point directly overhead, or **Zenith**, will be how far from the North Star?

Suppose him to travel in the same direction through 90°. What circle will he reach?
The point overhead will be how many degrees from the North Star?
Where, then, will the North Star be seen by an observer at the equator?
What is the latitude of the equator?

Suppose the observer to travel 10° north of the equator. How many degrees will the North Star be seen above the horizon?

How many degrees above the horizon will he see the North Star if he travels north from the equator

27 degrees?	23½ degrees?	2 degrees?
48 degrees?	66½ degrees?	90 degrees?

What relation, then, has the height of the North Star above the horizon at any point to the latitude of that point?
How, then, may we find the latitude of any point in the Northern Hemisphere?

NOTE. Learn the location of the North Star if you are not already acquainted with it. It will always be found nearly in line with the two end stars in the bowl of the Great Dipper, the most prominent group of stars in the northern sky.

* *

What is the altitude of the sun above the horizon at the equator at noon at the instant of either of the equinoxes?
If at a certain point north or south of the equator the sun at noon at this instant lacks 10° of being directly overhead, what must be the latitude of this point?

What will be the latitude of a point if at noon at this instant the sun is below the zenith

15 degrees?	45 degrees?	5 degrees?
40 degrees?	23½ degrees?	32 degrees?
48 degrees?	62½ degrees?	71 degrees?

How, then, on the day of either equinox will the distance of the sun from the zenith as seen at any point compare with the latitude of the given point?
Give, then, a second rule for determining the latitude of any point?

NOTE. An understanding of the preceding principles will make clear the process of determining the latitude and longitude of a ship at sea. The ship, evidently, must be provided (1) with a chronometer, set to the time of some standard meridian; (2) with a sextant, for determining the altitude of the sun or some other celestial body; (3) with tables showing the variation on each day of the year of the real course of the sun from the imaginary course which forms the basis of mean solar time; and (4) with tables showing the daily altitude of the sun above the horizon at some standard latitude.

112. Zones and Their Limits.

Observe carefully Diagram 4. At what latitude do the vertical rays from the sun strike the earth?

Suppose the earth to move around the sun from the equinoctial point until the northern half of its axis is inclined one degree towards the sun's rays. At how many degrees north of the equator will the vertical rays of the sun strike?

How many degrees north of the equator will the vertical rays of the sun strike if the northern half of the axis is inclined towards the sun's rays

5 degrees?	15 degrees?	20 degrees?
30 degrees?	17 degrees?	23 degrees?

The greatest inclination of the earth's axis from a perpendicular to the sun's rays is 23½ degrees, and the instant of this inclination, as we have learned, is called the summer solstice.

At the summer solstice, then, the vertical rays of the sun strike how many degrees north of the equator?

What would you say as to the sun's ever being seen directly overhead north of this latitude?

At the winter solstice what is the inclination of the northern end of the earth's axis from the sun's rays?

What, then, is the inclination of the southern end towards the sun's rays?

At how many degrees, then, south of the equator is the sun ever seen directly overhead?

Those belts of the earth in which the sun is seen directly overhead on some day of the year are called the **Torrid Zones.**

What is the northern limit of the torrid zone? the southern limit?

What is the width of the North Torrid Zone; of the South Torrid Zone? What is the total width of the torrid zone?

* *

Observe the earth in diagram 3. How many degrees does the sun shine beyond the North Pole?

How many degrees, then, from the North Pole will the sun

remain above the horizon during a complete rotation of the earth on its axis?

How many degrees from the South Pole will the sun remain below the horizon during a complete rotation of the earth on its axis?

Imagine a similar diagram to represent the position of the earth at the winter solstice.
At which pole will the sun during a complete rotation of the earth on its axis be above the horizon?
At which pole will it be below the horizon?

Those belts of the earth on which the sun remains above the horizon during a complete rotation of the earth on its axis are called the **Frigid** Zones. What is the width, measuring from a pole towards the equator, of each frigid zone?

* *

Those belts of the earth in which the sun is never seen directly overhead and in which the sun never remains above the horizon during a complete rotation of the earth on its axis are called the **Temperate** Zones.
What is the combined width of the North Frigid and the North Torrid Zones?
How many degrees is it from the North Pole to the equator?
What, then, is the width of the North Temperate Zone?
What is the width of the South Temperate Zone?

* *

Special names are given to the circles separating the different zones. Thus, the line forming the southern limit of the North Frigid Zone is called the **Arctic Circle**; the line forming the northern limit of the North Torrid Zone, the **Tropic of Cancer**; the line forming the northern limit of the South Frigid Zone, the **Antarctic** Circle; and the line forming the southern limit of the South Torrid Zone, the Tropic of **Capricorn.**

What is the latitude of the Tropic of Cancer? of the Tropic of Capricorn? of the Arctic Circle? of the Antarctic Circle?

In what zone is Vermont? Mexico? Cape Horn? Edinburgh? Rio de Janeiro? Chatanooga? Buenos Ayres? Cape of Good Hope? Florida?

NOTE. If difficulty is found in comprehending the cause of the constant variation in the angle between the axis of the earth and the line connecting the centre of the earth and the centre of the sun, make use of the following simple device:

Attach a string to one pencil at its centre and to a second pencil at a point a little above the centre.

Place the two pencils on end on a plane surface, holding the first pencil exactly vertical and inclining the second pencil towards the first about 23½ degrees.

Keeping the direction of the second pencil constantly parallel to its first direction, revolve it about the first pencil with the connecting string as a radius.

What is the greatest inclination at any point in the revolution of the upper half of the second pencil towards the string? from the string? At how many points do you find it to be perpendicular to the string?

At which point does the second pencil have the same direction relative to the string that the axis of the earth has relative to the sun's rays at the summer solstice? at the winter solstice? at either equinox?

118. Altitude of the Sun at Different Seasons.

Turn to Diagram 4. Where will the sun be seen at noon by an observer at the equator?

Suppose that he travels 5 degrees north. The point directly overhead will be how many degrees north of the sun?

The sun, then, will be how many degrees south of the zenith? how many degrees above the horizon?

How many degrees below the zenith and how many degrees above the horizon will the sun be seen if the observer travels north

23½ degrees?	20 degrees?	45 degrees?
66½ degrees?	75 degrees?	90 degrees?

Explain where the sun will be seen if the observer travels to the south each of the above number of degrees?

How, evidently, at the equinoxes does the distance of the sun at noon from the zenith compare with the latitude of the position of the observer?

What is the distance from the zenith to the horizon?

How, then, may we find for either equinox the height at noon of the sun above the horizon at any given latitude?

What at either equinox is the height at noon of the sun

At Buenos Ayres?	At Chatanooga?
At Cape of Good Hope?	At Edinburgh?
At Rio de Janeiro?	At Cape Horn?
In Vermont?	In Florida?

What at either equinox is the height at noon of the sun at the Tropic of Cancer? at the Tropic of Capricorn? at the Arctic Circle? at the North Pole? at your home?

In what direction is the sun seen at noon by observers north of the equator? by observers south of the equator?

Turn to Diagram 3. At what latitude do the vertical rays of the sun strike the earth?

At what latitude, then, will the sun be seen directly overhead?

What name is given to the circle including all points in this latitude?

How many degrees south of the zenith will the sun be seen at noon at a point north of the Tropic of Cancer

1 degree?	20 degrees?	66½ degrees?
10 degrees?	43 degrees?	89½ degrees?

Where will the sun be seen at noon with reference to the zenith at a point each of the above number of degrees south of the Tropic of Cancer?

Give a rule for finding for any latitude the distance of the sun from the zenith at noon at the summer solstice.

How after finding the distance of the sun from the zenith may we find its distance from the horizon?

What at noon at the summer solstice is the height of the sun above the horizon

At the Arctic Circle?	At the Antarctic circle?
At the equator?	At the Tropic of Capricorn?
At the North Pole?	At the South Pole?
At Buenos Ayres?	At Chatanooga?
At Edinburgh?	At Cape Horn?
At Mexico?	In Vermont?
In Florida?	At Cape of Good Hope?

* *

Imagine the earth's axis as represented in Diagram 3 to rotate about its centre and recede to the position occupied by it in Diagram 4. Imagine the upper half to continue to recede until it is inclined from the sun's rays 23½ degrees.

At what latitude will the vertical rays of the sun strike the earth?

Where, then, will the sun be seen at the Tropic of Capricorn?

How many degrees from the zenith will it be seen at a point any given number of degrees north or south of the Tropic of Capricorn?

Give a rule for finding the distance of the sun from the zenith at any latitude at the winter solstice.

Give a rule for finding its height above the horizon at the winter solstice.

Give the height of the sun at the winter solstice at the Tropic of Cancer; at the Arctic and at the Antarctic Circles; at the South Pole; at each locality given in Diagrams 3 and 4.

Ex. 1. Which is the higher at the summer solstice, the sun at the equator or the sun in Vermont?

SOLUTIONS.

(1)	(2)
23½°	21½°
21½°	21½°

Ex. 2. Employing the definitions of *torrid*, *frigid*, and *temperate rays* given on page 248, what rays does Vermont receive at the summer solstice? at the winter solstice?

EXPLANATIONS.

Ex. 1. At the summer solstice the sun is seen directly overhead at the Tropic of Cancer, or 23½ degrees north of the equator.

The equator is 23½ degrees south of the Tropic of Cancer, therefore at the equator the sun is 23½ degrees below the zenith. Vermont is only 45 less 23½, or 21½, degrees north of the Tropic of Cancer. The sun, therefore, as seen in Vermont is only 21½ degrees from the zenith, or 2 degrees higher than as seen at the equator.

Ex. 2. At the summer solstice the sun is seen in Vermont only 45 less 23½, or 21½, degrees from the zenith, therefore Vermont is then receiving torrid rays. At the winter solstice the sun is 45 plus 23½, or 68½, degrees from the zenith. It, therefore, is only 90 less 68½, or 21½, degrees from the horizon, and is emitting frigid rays. Vermont, therefore, receives in each year torrid and frigid as well as temperate rays.

NOTE. The following interesting facts concerning the apparent daily course of the sun and the relative length of the day and the night may be observed by the pupil who carefully studies the diagrams.

All lines connecting opposite points in the equator pass through the centre of the earth. Therefore, one half the equator is always in the sunlight, thus making the days and nights always equal; and the sun always rises exactly in the east and sets exactly in the west. At either equinox, people living at the equator see the sun pass directly overhead; from the vernal equinox to the summer solstice, they see it sinking farther each noon towards the northern horizon, and from the autumnal equinox to the winter solstice, farther each noon towards the southern horizon.

At the North Pole on the day of the Vernal Equinox, the sun pursues its course partly below and partly above the horizon. A week later its path is about two degrees above the horizon, and this altitude keeps increasing, until at the summer solstice it reaches a height of 23½ degrees. It then begins to retrace its course. It re-

turns to the horizon 6 months after it left it and slowly sinks below it, not to again become visible until a half-year later.

The variation in the relative length of the day and the night increases as a locality is more distant from the equator. Thus, in northern New England on June 22, 1902, the sun rose at 4 h. 11 m. a, m., and set at 7 h. 57 m. p. m.; while on Dec. 22 it will not rise until 7 h. 38 m. a. m., and will set at 4 h. 18 m. p. m.

What is the length of the day on June 22? of the night? of the day on Dec. 22? of the night

Ex. 121.

1. Find in your geography 5 cities in which the sun is at some time seen at the zenith; 5 in which it is seen within 20 degrees of the zenith; 5 in which it is at some time seen at noon at a distance of 15 degrees or less from the horizon.

2. Find to the nearest degree the least distance from the zenith ever reached by the sun at New Orleans; at Paris; at Athens; at Jerusalem; at Pekin; at Madrid; at Rome; at Melbourne.

Find the least distance from the horizon ever reached by the sun at noon at Sitka: at Montreal; at Copenhagen; at Christiana; at Pretoria.

3. Measure each week for 4 weeks or more the length of the shadow cast by some object, and find the ratio of each length to each preceding length.

4. Consult an almanac and from the data there given find the average length of the day for the month in which you are studying this subject.

114. The Metric System.

A decimal system called the **Metric System**, has been invented in France, and is now in use in that country and in most of the countries of Continental Europe.

Its basis is the **Meter**, which equals 39.37 inches and was intended to be one ten-millionth of the distance from the equator to the North Pole.

The use of this system was legalized in the United States in 1866. It is employed by writers on scientific subjects in all countries, and is so simple and complete that it seems incredible that it has not before this come into universal use.

Linear Measure. The meter is divided into decimeters, centimeters, and millimeters. Its multiples are dekameters, hektometers, kilometers, and myriameters. The prefixes *deci*, *centi*, and *milli* are from the Latin and signify tenths, hundredths, and thousandths; and the prefixes *deka*, *hecto*, *kilo*, and *myria* are from the Greek and signify tens, hundreds, thousands, and ten-thousands.

The scale throughout the table is ten. Give the table of linear measure in the metric system.

* *

Surface Measure. A surface a meter long and a meter wide contains how many square decimeters?

What, then, is the scale in surface measure?

The prefix *myria* is commonly employed only with linear measures and measures of weight. Omitting the denomination with this prefix, give the table of surface measure.

* *

Land Measure. In measuring land a square dekameter is taken as a unit and called an **Ar.**

Only one lower denomination, $\frac{1}{100}$ of an ar, and one higher denomination, 100 ars, are employed in connection with this unit. Moreover, while the units of the regular surface measure employ prefixes indicating the ratio of one of their sides to a side of the standard unit, the prefixes employed with the ar indicate the ratio of their surfaces to the surface of their standard unit. A hundredth of an ar, therefore, is called a centar, and a hundred ars a hectar.

* *

Solid Measure. A solid a meter long, a meter wide, and a meter thick contains how many cubic decimeters?

What, then, is the scale in solid measure?

In solid measure larger denominations than cubic meters are not commonly employed. Omitting these denominations, give the table of solid measure.

* *

Wood Measure. In measuring wood or stone, a cubic meter is called a **Ster.** The only lower denomination in common use is the decister, and the only higher denomination the dekaster. The prefixes are used to indicate the ratio of the volume of their units to the volume of the standard unit.

What, then, is the scale? Give the table of wood measure.

Measures of Capacity. In measuring liquids, grains, etc., a cubic decimeter is taken as a unit and called a **Liter.** The prefixes, as with wood measure, indicate the ratio of the volume of their units to the volume of the standard unit. All the prefixes are employed with this standard unit except *myria*.

What is the scale of this table? Give the table of measures of capacity?

* *

Measures of Weight. In measuring weight, the weight of a cubic centimeter of water at 39.2°, the temperature of its greatest density, is taken as a standard unit and called a **Gram.**

The scale of the table, as with linear measure, wood measure, and measures of capacity, is 10.

The prefix *myria* is employed in this table, and 10 myriagrams are called a **Quintal**, and 10 quintals a **Metric Ton.**

Give the table of measures of weight.

* *

If the pupil has mastered the relations between the standard units in the different tables of the common system of weights and measures, the ratio of the meter to the inch will be the only additional relation essential for converting units of one system into equivalent units of the other. It will be well, however, to fix in mind the following approximate relations.

A **Kilometer** is a little more than $\frac{3}{5}$ of a mile.

An **Ar** is a little less than $\frac{1}{4}$ of an acre.

A **Ster** is a little more than $\frac{1}{4}$ of a cord.

A **Liter** is a little more than 1 liquid quart, and a little more than $\frac{9}{10}$ of a dry quart.

A **Kilogram** is a little more than $2\frac{1}{5}$ pounds.

A **Metric Ton** is a little more than a long ton.

The units principally employed are the following:

In Linear Measure, the METER, CENTIMETER, MILLIMETER, KILOMETER; in Surface Measure, the SQUARE METER, SQUARE CENTIMETER, SQUARE KILOMETER; in Cubic Measure, the CUBIC METER, CUBIC CENTIMETER; in Measures of Capacity, the LITER, HEKTOLITER; and in Measures of Weight, the GRAM, MILLIGRAM, KILOGRAM, METRIC TON. Other denominations are commonly expressed in multiples or parts of these denominations, just as dimes are expressed as tens of cents.

SUMMARY OF DEFINITIONS.

Point. That which has only position.

Line. The path of a point.

Surface. The path of a line.

Solid. The path of a surface.

Angle. The difference in the direction of two lines.

Vertex of Angle. The point where the sides of an angle meet.

Perpendicular Line. A line meeting another line at right angles.

Unit of Surface. The path of a unit of length moving perpendicularly to itself through the same unit of distance.

Unit of Volume. The path of a unit of surface moving perpendicularly to itself through the corresponding unit of distance.

Parallelogram. The path of a line so moving that every position is parallel to every preceding position.

Rectangle. A parallelogram produced by a line moving perpendicularly to itself.

Rhomboid. A parallelogram produced by a line moving obliquely.

Square. An equal-sided recangle.

Rhombus. An equal-sided rhomboid.

Triangle. The path of a line that constantly and regularly diminishes until reduced to a point.

Trapezoid. The path of a line that constantly and regularly diminishes but that ceases its motion before being reduced to a point.

Quadrilateral. A four-sided figure.

Trapezium. A Quadrilatral having no two sides parallel.

Pentagon. A five-sided figure.

Perimeter. The line or lines bounding a surface.

Diagonal. A line connecting any two non-adjacent vertices of a polygon.

Circle. The path of a line revolving about one of its ends.

Radius. The line producing a circle.

Circumference. The path of the free end of a radius.

Centre. The position of the fixed end of a radius.

Diameter. Two radii so situated with reference to each other as to form one line.

Prism. The path of a polygon.

Cylinder. The path of a circle.

Pyramid. The path of a polygon that constantly and regularly diminishes until reduced to a point.

Cone. The path of a circle that constantly and regularly diminishes until reduced to a point.

Frustrum of a Pyramid. The path of a polygon that constantly and regularly diminishes, but that ceases its motion before being reduced to a point.

Frustrum of a Cone. The path of a circle that constantly and regularly diminishes, but that ceases its motion before being reduced to a point.

Sphere. The path of a semi-circle revolving about its diameter.

Similar Surfaces. Surfaces whose corresponding sides are proportional.

Similar Solids. Solids whose corresponding edges are proportional.

Parallelopipedon. The path of a parallelogram.

Rectangular Parallelopipedon. The path of a rectangle moving perpendicularly to itself.

Cube. The path of a square moving perpendicularly to itselt through a distance equal to the length of one of its sides.

Roll of Paper. A strip of paper 48 feet long and 18 inches wide.

Board Foot. Path of a square foot moving through an inch or less.

Thousand of Shingles. A quantity of shingles equivalent to 1000 shingles each 4 inches wide.

Bundle of Laths. 100 laths each 4 feet long and from 1 inch to 1⅜ inches wide.

Mean Solar Day. The average interval between consecutive instants at which the sun reaches its greatest daily altitude.

Tropical Year. The interval between two consecutive instants at which the earth's axis has the greatest inclination to the sun's rays.

Summer Solstice. The instant of the greatest inclination of the earth's axis towards the sun's rays.

Winter Solstice. The corresponding instant of least inclination.

Equinoxes. The two instants at which the earth's axis is perpendicular to the sun's rays.

Torrid Zones. Those belts of the earth in which on some day of the year the sun is seen directly overhead.

Frigid Zones. Those belts of the earth in which the sun for one or more days remains above the horizon.

Temperate Zones. Those belts of the earth in which the sun is never seen in the zenith, and in which it never remains above the horizon an entire day.

East. The direction towards which the earth is turning.

West. The direction from which the earth is turning.

Meridians. Circles passing around the earth through the poles.

Equator. A circle half way between the poles.

REVIEW QUESTIONS.

Define a point; a surface; a solid.

Name the standard units commonly used as measures of length. Give the table of linear measure.

Define a hand; a geographical mile. The equatorial circumference of the earth being given as 24899 miles, find the number of common miles in a geographical mile. Define a knot; a league; a fathom.

$15\frac{3}{4}$ hands are how many feet and inches? 2500 geographical miles are how many common miles? 2500 common miles are how many geographical miles? 26 knots are how many common miles? 48 leagues are how many common miles? 126 leagues are how many geographical miles? 60 feet are how many fathoms?

Give a rule for reducing a compound number to its lowest denomination; for reducing units of a lower denomination to a compound number of higher denominations; for reducing a fraction of a higher denomination to integers of lower denominations; for reducing a compound number to a fraction of a higher denomination; for adding denominate numbers; for subtracting denominate numbers; for multiplying denominate numbers; for dividing denominate numbers.

Define an angle; a right angle; a perpendicular.

Define a square inch; a square foot; a square yard; a square rod; a square mile; a square.

Define a rectangle. Give a rule for finding the area of a rectangle. Define a base; an altitude. Explain the process of finding the number of square inches in a square foot; of square feet in a square yard; of square yards in a square rod; of square feet in a square rod; of square rods in a square mile. What unit appears in square measure that has no corresponding unit in linear measure?

Define an acre. Form and give a table of square measure.

Define a chain. Explain the process of finding the number of square rods in a square chain. Form and give a table of surveyors' square measure.

Define parallel lines; a parallelogram; a rhomboid; a rhombus. Give a rule for finding the area of a parallelogram.

Define a trapezoid. Give a rule for finding the area of a trapezoid.

Define a triangle. Give a rule for finding the area of a triangle.

Define a quadrilateral; a trapezium. Give a rule for finding the

area of any quadrilateral not a trapezium; for finding the area of a trapezium.

Define a polygon; a pentagon; a hexagon; a heptagon; an octagon; a nonagon; a decagon; a regular polygon; a diagonal. Give a special rule for finding the area of a regular polygon.

Define a circle; a radius; a circumference; a diameter. Give a rule for finding the area of a circle.

Give a rule for finding a diameter when the radius is given; a radius when the diameter is given; a circumference when the radius is given; a radius when the circumference is given.

Define a cubic inch; a cubic foot; a cubic yard. Define a unit of volume.

Define a prism; a right prism; an oblique prism; a triangular prism; a parallelopipedon; a rectangular parallelopipedon; a cube.

Define the altitude of a prism; the edges; the faces; the convex surface; the ends.

Explain the process of finding the number of cubic inches in a cubic foot; of cubic feet in a cubic yard. Form and give a table of cubic measure.

Define a cylinder; a right cylinder; an oblique cylinder. Give a rule for finding the volume of any prism or cylinder.

Define a pyramid; a frustrum of a pyramid; a cone; a frustrum of a cone; an apex; slant height.

Give a rule for finding the volume of a pyramid or a cone.

Give a rule for finding the volume of a frustrum of a pyramid or a cone.

Define a sphere; Give a rule for finding the volume of a sphere.

Give a rule for finding the convex surface of a prism or a cylinder; of a pyramid or a cone; of a frustrum of a pyramid or a cone; of a sphere.

Give a rule for finding the convex surface of a prism or a cylinder; of a pyramid or a cone; of a frustrum of a pyramid or a cone.

Define similar surfaces; similar solids. Similar surfaces vary as what power of their corresponding dimensions? similar solids as what power of their corresponding dimensions?

In what form is paper manufactured and sold; Give a rule for finding the number of rolls of paper required for a given room.

In what form is carpeting manufactured? Give a rule for finding the quantity of carpeting required for a given room.

About how much mixed paint is required for 100 square feet of unpainted surface?

About how much lime is required for 100 feet of plastering?

What are the most common dimensions of bricks? Give a rule for finding the quantity of bricks required for a given chimney or wall.

What are the dimensions of a perch of stone or masonry?

What is the unit of wood measure? A cord is how many cubic feet? What name is given to $\frac{1}{8}$ of a cord; Give a general rule for finding the number of cords in a pile of wood of any given dimensions; in a pile of four-foot wood of any given length and height; in a pile of four-foot wood 4 feet high and of any given length.

What is the unit of measurement for rectangular lumber? Explain the signification of the term "board foot." Give a general rule for finding the number of board feet in a stick of rectangular lumber of any given dimensions; in a stick 12 feet long and of any given width and thickness; in a board 12 feet long, 1 inch or less thick, and of any given width.

Explain the process of finding the quantity of matched boards required for a given surface.

By what two units are clapboards measured? Give a rule for finding the number of clapboards required for a given surface; for finding the number of feet of clapboards required for a given surface.

By what unit are laths sold? Give a rule for finding the quantity of laths required for a given surface.

By what unit are shingles sold? Give a rule for finding the quantity of shingles required for a given surface.

Define money. What are the principal divisions of Uuited States Money? Name the coins that are unlimited legal tender; that are legal tender for $10; that are legal tender for 25 cents.

Give the two principal divisions of paper money. Which of the notes issued by the Government are legal tender? Which are not legal tender? In what year were the United States Notes issued? the Treasury Notes? the Gold Certificates? the Silver Certificates?

Are Bank Notes legal tender? What one restriction is there to the legal tender quality of the Treasury Notes?

What was the original unit of value in the United States? When was it adopted? How many grains of pure silver was it made to contain? The weight of the gold in the gold eagle was made what part of the weight of the silver in ten silver dollars? What is the present weight of the silver in the silver dollar? of the gold in the gold dollar? What is the present ratio of the weight of the silver in ten silver dollars to the weight of the gold in the gold eagle?

What is said concerning the ratio of the weight of the silver in the minor silver coins to the weight of the silver in the silver dollar?

What is the unit of English money? of French money? of German money? Give the table of each of the three kinds of money.

What are the standard units in the common system of measures of capacity? of measures of weight? What criticism, if any, would you make of this system? Give the table of liquid measure; of dry measure; of troy weight; of avoirdupois weight; of apothecaries' weight; of apothecaries' measure.

What is the unit of time? Define this unit. Why does it not equal the time of the rotation of the earth on its axis? Why is not the time from noon to noon, reckoned by the actual course of the sun, always the same?

What is the second unit of time? Define this unit. Define a civil year; a common year; a leap year. Define the summer solstice; the winter solstice; the vernal equinox; the autumnal equinox. Give the rule for determining a leap year. Give the table of time measure; the number of days in each month.

What is the time from Jan. 9 to May 15? from Jan. 9 to June 4? from Jan. 9 to Dec. 2? from Apr. 9, 1901 to Feb. 6, 1905? Explain each solution.

Give the table of angular measure; of miscellaneous measures. Complete and read the tables following the table of miscellaneous measures.

Define the earth's axis; the North Pole; the South Pole; the equator; meridians; latitude; longitude. Give a rule for finding the difference in the latitude of two places; for finding the difference in the longitude of two places.

Define east; west. When it is noon at Greenwich what time is it 15 degrees east of Greenwich? 15 degrees west of Greenwich?

When it is Monday noon at the 180th meridian what time is it 15 degrees west of that meridian? 15 degrees east? Give a rule for finding the difference in the time of two places when the difference in longitude is given; for finding the difference in longitude when the difference in time is given. Explain the system of Standard Time.

What is the inclination of the earth's axis? Define the Torrid Zones; the Frigid Zones; the Temperate Zones. Explain the process of finding the latitude of a place from the altitude of the North Star; of the Sun. How may we find the altitude of the North Star? of the Sun at either equinox;

Give the tables of the Metric System; the standard units.

REVIEW EXERCISES.

1. Add 3 gal. 2 qt. 1 pt., 7 gal. 3 qt. 1 pt., 9 gal. 3 qt., 6 gal. 1 pt., 2 gal. 1 pt., 2 qt. 3 gi., 1 pt. 2 gi., and 5 gal. 3 qt. 1 pt. 3 gi.

2. Subtract 10 bu. 2 pk. 6 qt. 1 pt. from 21 bu. 3 pk. 4 qt.

3. Multiply 15 mi. 75 rd. 1 yd. 2 ft. 11 in. by 25.

4. Divide 135 sq. mi. 250 A. 96 sq. rd. 25 sq. yd. 7 sq. ft. 100 sq. in. by 16.

5. Reduce 74 cu. yd. 24 cu. ft. 1429 cu. in. to cubic inches.

6. Reduce 24389″ to units of higher denominations.

7. Reduce $\frac{7}{8}$ of a leap year to integers of lower denominations.

8. Reduce .257 of a pint to the integers of lower denominations in Apothecaries' Measure.

9. Reduce 10 oz. 13 pwt. 21 gr. to a fraction of a pound; to a decimal of a pound.

10. A rectangle is 3 ft. 5 in. long, and 2 ft. 9 in. wide. Find its area in square feet.

11. The base of a parallelogram is $7\frac{5}{8}$ inches, and the altitude $4\frac{5}{16}$ in. Find its area.

12. The base of a triangular field is 57 rd. 12 ft., and the altitude 39 rd. 8 ft. Find the number of acres in the field.

13. The bases of a field in the form of a trapezoid are respectively 43.86 and 37.53 chains, and the altitude is 25.29 chains. Find the number of acres in the field.

14. The diagonal of a trapezium is 3 ft. 8 in., and the perpendiculars let fall upon it from the opposite angles are respectively 3 ft. 9 in. and 5 ft. 7 in. Find the area of the trapezium in square feet and square inches.

15. Each side of a regular hexagon is 12 inches, and the perpendicular let fall upon each side from the centre is 10 ft. $4\frac{5}{8}$ in. How many square feet are there in the hexagon?

16. The radius of a circle is $7\frac{9}{16}$ inches. What is its surface?

17. The circumference of a circle is 3 ft. $5\frac{7}{8}$ in. How many square feet and square inches does it contain?

18. The dimensions of a right prism are 4 ft. 7 in., 3 ft. 2 in., and 2 ft. 5 in. What are its contents in cubic feet?

19. The circumference of the base of a cylinder is 2 ft. 9 in.,

and the altitude of the cylinder is 5 ft. 11 in. What is its volume in cubic feet and cubic inches?

20. The sides of the bases of a rectangular pyramid are 14 ft. 9 in. and 12 ft. 5 in., and the altitude of the pyramid is 17 ft. 8 in. How many cubic feet are there in the pyramid?

21. The circumference of the base of a cone is 3 ft. 8 in. and the altitude of the cone is 6 ft. 7 in. What is the volume of the cone in cubic feet?

22. The base of the larger end of a triangular pyramid is 2 ft. 5 in. and the altitude 3 ft. 4 in., and the base of the smaller end is 1 ft. 9 in. and the altitude 2 ft. 4 in. The altitude of the frustrum is 3 ft. 7 in. What is its volume in cubic feet aud cubic inches?

23. The diameter of the larger end of the frustrum of a cylinder is 4 ft. 7. in. and of the smaller end 3 ft. 8 in. The altitude of the cylinder is 7 ft. 9 in. What is its volume in cubic feet?

24. The circumference of a sphere is $7\frac{5}{16}$ inches. What is its volume?

25. The dimensions of a right prism are 6 ft. 7 in., 3 ft. 2 in., and 2 ft. 5 in. What is its entire surface in square feet?

26. The diameter of the base of a right cylinder is 1 ft. $3\frac{3}{8}$ in., and the altitude of the cylinder is 2 ft. 11 in. What is its entire surface in square feet and square inches?

27. The dimensions of the base of a right pyramid are $10\frac{5}{8}$ and $7\frac{3}{4}$ inches, and the slant height of the pyramid is 2 ft. $3\frac{1}{2}$ in. What is its convex surface in square feet?

28. The diameter of the base of a right cone is $9\frac{3}{8}$ inches, and the slant height of the cone is 1 ft. $9\frac{1}{2}$ in. What is its entire surface?

29. The dimensions of the larger end of a frustrum of a right pyramid are respectively 33 and 22 inches, and of the smaller end 9 and 6 inches. The slant height of the frustrum is 20 inches. What is its convex surface?

30. The diameter of the larger end of a frustrum of a cone is $9\frac{3}{8}$ inches and of the smaller end $7\frac{3}{4}$ inches. The slant height of the frustrum is $23\frac{3}{8}$ inches. What is its entire surface?

32. The diameter of a sphere is 2 ft. $9\frac{11}{16}$ in. What is its surface in square feet and square inches?

33. The value of a log 18 inches in diameter is \$1.10. What is the value of a second log 40 inches in diameter?

34. A room is 16 feet long and 14 feet wide. Its height from the base-board, which is 8 inches wide, is 8 ft. 4 in. It has 3 doors, each 7 ft. 2 in. by 3 ft. 8 in., and 3 windows, each 6 ft. 4 in. by 3 ft. 6 in.

a. Find approximately the quantity of paper and of border required for the room, the width of the border being 18 inches.

b. Find approximately the quantity of laths required for the room.

c. Assuming the waste in matching to be 8 inches, find the quantity of carpeting 1 yard wide required for the room; the quantity of carpeting $\frac{3}{4}$ of a yard wide.

d. Find the quantity of planed and matched boards required for the floor of the room, each board being 4 inches wide, and $\frac{1}{2}$ inch of each board being wasted in planing and matching.

35. A building is 40 feet long and 30 feet wide. The height of the sides of the building is 22 feet, and the additional height to the gable ends is 12 feet. The length of the rafters is 19 ft. 3 in., and the width of the jet is 16 inches.

a. Find the quantity of boards required for the sides, ends, and roof of the house.

b. Find the quantity of shingles required for the house, assuming the shingles to be laid $5\frac{1}{2}$ inches to the weather.

c. Find the quantity of clapboards required for the house, assuming each clapboard to be 6 inches wide and to be laid $3\frac{3}{4}$ inches to the weather.

d. Find approximately the number of gallons of paint required for the building.

36. Find the number of cords in a pile of 16-inch wood 25 ft. 7 in. long and 8 ft. 6 in. high.

37. What is the time from March 17, 1901 to Jan. 10, 1905?

38. The difference in the longitude of two places is 37° 49′ 38″. What is the difference in time.

39. The difference in the time of two places is 4 h. 22 min. 41 sec. What is their difference in longitude.

40. The latitude of a certain town in Massachusetts is 42′ 34″. What is the greatest altitude of the sun?

Problems in Specific Gravity and Astronomy.

For the data required in solving these problems see the tables following the explanations.

SOLUTIONS.

Ex. 1. The base of the Great Pyramid is 764 ft. square. Its altitude is 480 ft. 9 in. It is composed of granite whose specific gravity is about 2.72. What is its weight in tons?

(1)

$$764\times764\times\frac{1923}{4}\times\frac{1}{3}\times\frac{125}{2}\times2{\cdot}72\times\frac{1}{2000}$$

```
   764
   6876
      764
  145924
  1313316
    291848
     437772
  2806111852
 16836711112
  22448948816
 190816059336
  63605353312
   7950669·14
```

Ex. 2. The mean diameter of the Earth is 7913 miles, and of Mars 4920 miles. What is the ratio of the volume of the Earth to the volume of Mars?

(2)

```
          7913                          4920
          23739                     24600000
        71217                         393600
      55391                         24206400
      626155569                 121032000000
      187846707                   1936512000
    563540121                   119095488000
  438308983
  4954769997497  4·2
  4763819520000
   190950454 97
```

Ex. 3. The inside dimensions of a tank are 3.57 meters. 2.5 meters, and 1.11 meters. How much water will it contain and what will be its weight?

(3)

3.9·627
9.9·0675
9906.75

EXPLANATIONS.

Ex. 1. The volume of the pyramid is 764 times 764 times $\frac{1923}{4}$ times $\frac{1}{3}$. To find the weight of this volume of water we multiply by $62\frac{1}{2}$, or $\frac{125}{2}$. To find the weight of the same volume of granite we multiply the weight of the water by 2.72, the specific gravity of granite, and to express this weight in tons we divide by 2000. An expression for the weight of the pyramid, therefore, is $764\times764\times\frac{1923}{4}\times\frac{1}{3}\times2.72\times\frac{1}{2000}$.

The operations that we perform are, (1) multiplying 764 by $\frac{1}{4}$ of 764, or by 191; (2) multiplying this product by 1923; (3) multiplying our second product by $\frac{1}{4}$ of 272, or by 68; (4) dividing our third product by 3; (5) dividing this quotient by $\frac{1}{125}$ of 1000, or by 8; and (6) pointing off two decimal places on account of our third multiplier's being a decimal instead of an integer, The result thus obtained is 7,950,669.14 tons.

Explain the steps used in the multiplying by 191; by 1923; by 68.

Ex. 2. Every volume is the product of three factors; therefore, similar volumes vary as the cubes of their corresponding dimensions.

The ratio of the diameter of the Earth to the diameter of Mars is $\frac{7913}{4920}$. The ratio of the volume of the Earth to the volume of Mars, therefore, is $(\frac{7913}{4920})^3$, or 4.2.

Ex. 3. The volume of the tank is 2.5, or ¼ of 10 times 1.11 times 3.57, or 9.90675, cubic meters

A liter is a cubic decimeter. To change cubic meters to liters, therefore, we must multiply by 1000, or remove the decimal point 3 places to the right. But to change our product to hektoliters, we must divide by 100, or move our decimal point 2 places to the left. In place of these two operations, we simply remove the decimal point one place to the right, and obtain as the contents of the tank 99.0675 hektoliters or 99 hektoliters 6.75 liters.

A cubic centimeter of water weighs a gram, and 1000 cubic centimeters, or a cubic decimeter, weighs 1000 grams, or a kilogram. The weight of the water, therefore, is 1000 times 9.90675, or 9906.75, kilograms,

Explain the multiplication of 3.57 by 111.

TABLE OF SPECIFIC GRAVITIES.

Air	.0013	Aluminum	2.06
Cork	.24	Slate	2.11
White Pine	.40	Salt	2.13
Spruce	.48	Graphite	2.30
Mahogany	.64	Marble	2.72
Maple	.66	Granite	2.72
Ash	.67	Glass	2.76
Birch	.69	Diamond	3.50
Alcohol	.79	Zinc	7.
Lime	.80	Tin	7.28
Oak	.83	Brass	7.61
Ice	.93	Iron	7.80
Distilled Water	1.	Steel	7.82
Milk	1.03	Nickel	8.41
Sea Water	1.03	Copper	8.95
Lignum Vitæ	1.33	Silver	10.50
Anthracite Coal	1.50	Lead	11.36
Sand	1.65	Mercury	13.58
Sulphur	2.	Gold	19.26
Brick	2.05	Platinum	21.60

THE PRINCIPAL PLANETS.

PLANETS.	MEAN DISTANCES FROM THE SUN IN MILES.	MEAN DIAMETERS IN MILES.	LENGTH OF YEAR IN DAYS.	LENGTH OF DAYS IN HOURS AND MINUTES.
Mercury,	35,352,000	2,900	88	24 5
Venus,	66,131,500	7,510	225	23 21
Earth,	91,430,220	7,913	365	23 56
Mars,	139,312,200	4,920	687	24 37
Jupiter,	475,693,100	88,390	4,333	9 56
Saturn,	872,134,600	71,900	10,759	10 29
Uranus,	1,753,851,000	33,000	30,647	9 39
Neptune,	2,746,271,200	36,000	60,127	....
(Sun),	....	866,400	....	....

1. Find the rate, in miles a minute, at which a person at the equator moves through space in consequence of the rotation of the Earth on its axis. Find the rate at which a person moves at the equator of Mercury; of Venus; of Mars; of Jupiter; of Saturn; of Uranus.

2. Find the distance passed over by a person while accompanying the Earth on its yearly circuit about the Sun. Find how long it would take him to travel the same distance on a train moving at the rate of 40 miles an hour.

3. Find the ratio of the surface of the United States to the surface of Jupiter; of the surface of North America to the surface of Mars; of the surface of the Earth to the surface of the Sun.

4. What would be the quantity of gold, measured in quarts or bushels, that would weigh 100 avoirdupois pounds? What would be the quantity of silver? of iron? of granite? of salt? of sand? of anthracite coal? of ice?

5. Find the weight of 1000 board feet of white pine lumber, at least 1 inch thick; of 1000 board feet of spruce; of mahogany; of maple; of ash; of birch; of oak.

6. What is the weight of a column of granite 21 inches in diameter and 18 feet high? of a cubic yard of sand? of 1000 bricks of standard size? of the air in a room 20 feet long, 16 feet wide, and 10 feet high?

7. Suppose the inclination of a planet's axis to be 40 degrees. What will be the width of its Torrid Zones? of its Frigid Zones? of its Temperate Zones?

ORIGINAL PROBLEMS.

1. Form and solve two problems in mensuration making the units in each a hand and an inch; making the units a geographical and a common mile; a fathom and a foot.

2. Form and solve an exercise in reducing a compound number to its lowest denomination; in reducing units of a lower denomination to a compound number of higher denominations; in reducing a fraction of a higher denomination to integers of lower denominations; in reducing a compound number to a fraction of a higher denomination; in adding denominate numbers; in subtracting denominate numbers; in multiplying denominate numbers; in dividing denominate numbers.

3. Make and solve two problems in finding the area of a rectangle; of a rhomboid; of a rhombus; of a square; of a trapezoid; of a triangle; of an irregular polygon; of a regular polygon; of a circle.

4. Make and solve two problems in finding the volume of a prism; of a cube; of a cylinder; of a pyramid; of a cone; of a frustrum of a pyramid; of a frustrum of a cone; of a sphere.

5. Make and solve a problem in finding the entire surface of a pentagonal prism; of a cylinder; of a pyramid; of a cone; of a frustrum of a pyramid; of a frustrum of a cone.

6. Make and solve a problem in papering; in carpeting; in painting; in plastering; in brick-work; in measuring wood; in measuring rectangular lumber; in measuring shingles; in measuring laths.

7. Make and solve a problem in changing liquid quarts into dry quarts; dry quarts into liquid quarts; in changing troy pounds into avoirdupois pounds; avoirdupois pounds into troy pounds.

8. Make and solve five problems in finding the time in years, months, and days between two dates.

9. Make and solve a problem in finding the difference in longitude when the difference in time is given; the difference in time when a difference in longitude is given.

10. Make and solve a problem in finding the altitude of the sun when the latitude of a place is given; in finding the latitude of a place when the altitude of the sun is given.

Interest.

A loans B \$100. At the end of a year B returns the money with 6% of \$100, or \$6, additional for its use.

In business language money paid for the use of money is referred to as **Interest**, the money loaned as the **Principal**, the number of hundredths of the money paid for its use for some standard unit of time, usually a year, as the **Rate**, and the total money returned as the **Amount**.

What is the principal in the preceding transaction? the rate? the interest? the amount?

Define interest, principal, rate, amount. Define interest as a process.

115. To Find the Interest on \$1 at 6 Per Cent.

One of the most common rates at which money is loaned is 6%. Moreover, when the rate is other than 6% it is usually most convenient to first find the interest at 6% and then make the necessary correction.

NOTE. In ordinary interest computations a month is considered to be 30 days.

What is the interest at 6% on \$1 for 1 year? for 12 months?

Two months are what part of 12 months? What, then, is the interest on \$1 for 2 months? for 60 days?

Six days are what part of 60 days? What, then, is the interest on \$1 for 6 days?

One month is what part of 2 months? What, then, is the interest on \$1 dollar for 1 month? What is the interest expressed in mills?

One day is what part of 6 days? What, then, is the interest on \$1 for 1 day?

The facts that we have discovered, which form the basis for all interest computations at 6%, are expressed in the following table:

Interest on $1 at 6 per cent for

1 year = 6 cents.	**1 month = 5 mills.**
2 months = 1 cent.	**1 day = 1-6 mill.**
6 days = 1 mill.	

116. Interest on Any Principal at 6 Per Cent for Any Number of Days.

We wish to find the interest on $934 for 60 days.
What is the interest on $1 for 60 days?
One cent is what part of $1?
The interest on any principal for 60 days is, evidently, what part of the principal?
How do we find $\frac{1}{100}$ of a number?
What, then, is the interest on $934 for 60 days?

Without written work name the interest for 60 days on each of the following principals:

$175.	$1034.	$3578.	$325.39.
508.	8363.	4376.	875.56.
84.	976.	942.	934.01.
418.	2693.	7310.	573.70.

Give a special rule for finding the interest on any principal for 60 days.

* *

We wish to find the interest on 756.36 for 6 days.
What is the interest on $1 for 6 days?
One mill is what part of a dollar?
The interest on any principal for 6 days is, evidently, what part of the principal?
How do we find $\frac{1}{1000}$ of a number?
What, then, is the interest on $756.36 for 6 days?

Without written work name the interest for 6 days on each of the following principals:

$9714.	$589.	$2304.	$447.86.	$25.97.
376.	304.	978.	96.24.	10.97.
415.	563.	35.	6.35.	68.69.

Give a special rule for finding the interest on any principal for 6 days.

* *

What part of 60 days are

30 days?	12 days?	5 days?
20 days?	10 days?	4 days?
15 days?		

Give a special rule for finding the interest on any principal for

30 days.	12 days.	5 days.
20 days.	10 days.	4 days.
15 days.		

Using the pencil or crayon only to write in order the figures of the final result, find the interest at 6% on

\$3256 for 15 days.	\$934.16 for 4 days.
750 for 30 days.	35.34 for 10 days.
984 for 20 days.	738.96 for 5 days.
2847.76 for 12 days.	

* *

What part of 6 days are

3 days?	2 days?	1 day?

Give a special rule for finding the interest on any principal for

3 days.	2 days.	1 day.

Using the pencil or crayon only as before directed find the interest on

\$837.26 for 3 days.	\$2340 for 2 days.
74.56 for 1 day.	

NOTE. Make and solve problems like the preceding until you can do your work without hesitation and with absolute accuracy.

* *

We wish to find the interest on a certain principal for 90 days.

90 equals 60 plus what?

30 is what part of 60?

How, then, can we find the interest on any principal for 90 days?

We wish to find the interest on a certain principal for 45 days?

45 equals 60 less what?

15 is what part of 60?

How, then, can we find the interest on any principal for 45 days?

How can we find the interest on any principal for 60 days plus or minus a convenient part of 60 days?

Without written work find the interest

On \$20 for 80 days.	On \$160 for 90 days.
On \$40 for 50 days.	On \$90 for 40 days.
On \$45 for 56 days.	On \$72 for 65 days.
On \$20 for 57 days.	On \$35 for 72 days.
On \$70 for 64 days.	On \$48 for 55 days.

Ex. 1. Find the interest on $93.56 for 63 days.

Ex. 2. Find the interest on $784.36 for 123 days.

Ex. 3. Find the interest on $2016.24 for 93 days.

Ex. 4. Find the interest on $834.29 for 17 days.

Ex. 5. Find the interest on $135.24 for 3 days.

Ex. 6. Find the interest on $76.24 for 38 days.

SOLUTIONS.

(1)	(2)	(3)
$·93.56	$7·84.36	$20·16.24
·04 68	.39 22	10.08 12
		1.00 81
$.98	$16.08	$31.25

(4)	(5)	(6)
$1.6686	$.07	$.07624
.6952		.02541
$2.36		$.48

EXPLANATIONS.

Ex. 1. The interest on any principal for 60 days is one cent on each dollar, or .01 of the principal. Therefore, the interest on $93.56 for 60 days is .01 of $93.56, or $.9356.

The interest on $93.56 for 3 days is $\frac{1}{20}$ its interest for 60 days, or $\frac{1}{20}$ of $.9356, or $.0468. Uniting these two interests, we find the interest for the total time to be $.98.

Ex. 2. The interest on $784.36 for 60 days is $7.8436, and for 3 days $\frac{1}{20}$ of $7.8436, or $.3922. To find the interest for 123 days, we multiply $7.8436 by 2, and, as we multiply, combine our product with $.3922, the interest for 3 days. We thus obtain as the interest for the total time $16.08.

Ex. 3. The interest on the given principal is $20.1624 for 60 days, $10.0812 for 30 days, and $1.0081 for 3 days. The total interest is the sum of these three interests, or $31.25.

Ex. 4. The interest on the given principal for 60 days is $8.3429. For 12 days it is $\frac{1}{5}$ of $8.3429, or $1.6686, for 5 days $\frac{1}{12}$ of $8.3429, or $.6952, and for 17 days the sum of $1.6686 and $.6952, or $2.36.

Ex. 5. The interest on any principal for 6 days is one mill on each dollar, or .001 of the principal. Therefore, the interest on $135.24 for 6 days is .001 of $135.24, or $.13524.

The interest on $135.24 for 3 days is $\frac{1}{2}$ its interest for 6 days, or $\frac{1}{2}$ of $.13524, or $.07.

Ex. 6. The interest on $76.24 for 6 days is $.07624, and for 2 days $\frac{1}{3}$ of $.07624, or $.02541. To find the interest for 38 days, we multiply $.07624 by 6, and, while multiplying, combine our prod-

uct with $.02541, the interest for 2 days. We thus obtain as the total interest $.48.

NOTE 1. Observe that the creditor is given the fraction of a cent when it is exactly ½. Thus, the interest on $450 for 45 days is $4.50 less $1.12, instead of $4.50 less $1.13.

Observe that in taking $\frac{1}{20}$ of $7.843 we divide by 2 and move each quotient figure one place to the right.

Explain the process of taking $\frac{1}{200}$ of a number; $\frac{1}{2000}$.

NOTE 3. What, in your opinion, is the most convenient way of finding the interest on any principal for

8 days?	13 days?	17 days?	23 days?
10 days?	14 days?	19 days?	26 days?
11 days?	16 days?	21 days?	28 days?

Ex. 122.

Find the interest on

1. $483 for 63 days.
2. $956.75 for 90 days.
3. $1200 for 123 days.
4. $354 for 60 days.
5. $75 for 93 days.
6. $90 for 120 days.
7. $250 for 30 days.
8. $94.76 for 15 days.
9. $296 for 33 days.
10. $328.76 for 18 days.
11. $950 for 45 days.
12. $432 for 5 days.
13. $916.25 for 4 days.
14. $85 for 2 days.
15. $316.24 for 3 days.
16. $2346 for 6 days.
17. $293.75 for 1 day.
18. $93.26 for 57 days.
19. $306 for 36 days.
20. $98.76 for 11 days.
21. $150 for 113 days.
22. $75.26 for 47 days.
23. $350 for 19 days.
24. $975 for 35 days.
25. $378.34 for 23 days.
26. $59.39 for 17 days.
27. $142.25 for 8 days.
28. $900 for 24 days.
29. $536 for 32 days.
30. $12854 for 67 days.

117. To Find the Interest on $1 For Any Number of Years, Months, and Days.

We wish to find the interest on $1 for 3 yr. 6 mo.
What is the interest on $1 for 3 years? for 6 months?
What, then, is the interest for 3 yr. 6 mo.?

We wish to find the interest on $1 for 5 yr. 8 mo. 12 da.
What is the interest on $1 for 5 years? for 8 months? for 5 yr. 8 mo.? for 12 days?
What, then, is the interest for 5 yr. 8 mo. 12. da. ?

We wish to find the interest on $1 for 4 yr. 5 mo. 24 da.
What is the greatest even number less than 5? 5 less 4 equals what?
What is the interest on $1 for 4 yr. 4 mo.? for 1 mo. 24 da.?
What, then, is the interest for 4 yr. 5 mo. 24 da?

We wish to find the interest on $1 for 7 yr. 11 mo. 28 da.
What is the greatest even number less than 11?
What is the greatest multiple of 6 less than 28?
11 less 10 equals what?
28 less 24 equals what?
What is the interest on $1 for 7 yr. 10 mo.? for 1 mo. 24 da.? for 4 da.?
What, then, is the interest on $1 for 7 yr. 11 mo. 28 da.

NOTE. Observe that to solve problems like the last it is necessary to proceed in the following order:

1. To write down the interest for the years and even months.
2. To write down the interest for the odd month, if any, and the highest multiple of 6 days.
3. To write down the interest for the remaining days if any.

	SOLUTION.		
	(1)	(2)	(3)
Ex. 1. Find the interest at 6% on $1 for 1 yr. 2 mo. 6 da.	.071	.33¼	.217½

Ex. 2. Find the interest at 6% on $1 for 5 yr. 6 mo. 15 da.

Ex. 3. Find the interest at 6% on $1 for 3 yr. 7 mo. 15 da.

EXPLANATION.

Ex. 1. The interest on $1 for 1 yr. 2 mo. is 6 cents plus 1 cent, or 7 cents. The interest on $1 for 6 days is 1 mill. Therefore, the interest on $1 for 1 yr. 2 mo. 6 da. is 7 cents 1 mill.

Ex. 2. The interest on $1 for 5 yr. 6 mo. is 30 cents plus 3 cents, or 33 cents. The interest on $1 for 15 da. is $\frac{15}{60}$, or ¼, cent. Therefore, the interest on $1 for 5 yr. 6 mo. 15 da. is 33¼ cents.

Ex. 3. We are to find the interest on $1 for 3 yr. 7 mo. 15 da.

7 months equal 6 months plus 1 month, and 15 days equal 12 days plus 3 days.

The interest on $1 for 3 yr. 6 mo. is 18 cents plus 3 cents, or 21 cents; the interest on $1 for 1 mo. 12 da. is 5 mills plus 2 mills, or 7 mills; and the interest on $1 for 3 days is ½ mill. Therefore, the interest on $1 for 3 yr. 7 mo. 23 da. is $.217½

NOTE. Practise upon exercises like the preceding until you can solve them without the least hesitation and with absolute accuracy.

Ex. 123.

Give the interest at 6% on $1 for

1.	7 yr.	11.	5 yr. 9 mo. 25 da.
2.	5 yr. 10 mo.	12.	4 yr. 7 mo. 18 da.
3.	6 yr. 8 mo. 18 da.	13.	7 yr. 11 mo. 8 da.
4.	3 yr. 6 mo. 6 da.	14.	9 yr. 10 mo. 8 da.
5.	8 yr. 2 mo. 24 da.	15.	3 yr. 8 mo. 15 da.
6.	3 yr. 7 mo. 12 da.	16.	5 yr. 7 mo. 27 da.
7.	5 yr. 5 mo. 18 da.	17.	4 yr. 6 mo. 16 da.
8.	7 yr. 7 mo. 12 da.	18.	3 yr. 3 mo. 15 da.
9.	2 yr. 3 mo. 6 da.	19.	6 yr. 0 mo. 5 da.
10.	1 yr. 1 mo. 24 da.	20.	2 yr. 5 mo. 26 da.

118. To Find the Interest at 6 Per Cent on Any Principal for Any Time.

We find the interest on $1 for a given number of years, months, and days to be a certain amount. How can we find the interest on any number of dollars for the same time?

Give a rule for finding the interest at 6% on any principal for any time.

Ex. 1. Find the interest at 6% on $976 for 4 yr. 5 mo. 14 da.

Ex. 2. Find the interest at 6% on $435.26 for 6 yr. 5 mo. 25 da.

Ex. 3. Find the interest at 6% on $560 for 2 yr. 9 mo. 22 da.

Ex. 4. Find the interest at 6% on $96.64 for 5 yr. 6 mo. 15 da.

Ex. 5. Find the interest at 6% on $780 for 2 yr. 8 mo. 20 da.

Ex. 6. Find the interest at 6% on $543.16 for 5 yr. 8 mo. 10 da.

SOLUTIONS.

(1)
$976
.267⅓
·325
6 832
58 56
195 2
$260.92

(2)
$174.104
4.3526
169.7514
.3627
$169.39

(3)
$560 × .168⅔
4.480
373
89.60
$94.45

(4)
$2.8992
31.8912
.2416
$32.13

(5)
$124.80
2.60
$127.40

(6)
$543.16
.34$\frac{1}{6}$
·9053
21 7264
162 948
$185.58

EXPLANATIONS.

Ex. 1. The interest on \$1 for 4 yr. 5 mo. 14 da. is \$.267$\frac{1}{3}$, and on \$976, .267$\frac{1}{3}$, times 976, or 260.92, dollars.

Ex. 2. We first find the interest on \$435.26 for 6 yr. 6 mo. This interest is 39, or 40 less 1, times .01 of \$435.26, or \$169.7514. From this interest we deduct \$.3627, the interest for 5 days. We thus find the interest on the given principal for the given time to be \$169.39.

Ex. 3. The interest on \$1 for 2 yr. 9 mo. 22 da. is \$.168$\frac{2}{3}$, and on \$560, .168$\frac{2}{3}$, or .168$\frac{8}{12}$, times 560, or 94.45, dollars.

Ex. 4. We first find the interest for 5 yr. 6 mo. To this interest, which is .33, or 11 times 3 times .01, times \$96.64, we add \$.2416, the interest for 15 days. We thus find the total interest to be \$32.13.

Ex. 5. The required interest is \$124.80, the interest for 2 yr. 8 mo., plus \$2.60, the interest for 20 days, or \$127.40.

Ex. 6. Explain the solution of this exercise.

NOTE. Observe the following points in the preceding solutions.

1. No common fractions appear in the partial products. Thus, 9053 is written as $\frac{1}{6}$ of 54316.

2. The decimal point is expressed in the first partial product, and only the cents are written in the final product, allowance, of course, being made for the sum of the lower denominations.

Ex. 124.

Find the interest at 6 per cent on

			YR.	MO.	DA.				YR.	MO.	DA.
1.	\$324.75	for	3	7	13	16.	\$214	for	2	8	20
2.	208	"	5	8	21	17.	316.25	"	3	9	17
3.	307	"	2	9	7	18.	984	"	5	6	25
4.	450	"	4	3	16	19.	434.26	"	3	3	27
5.	287.36	"	7	11	15	20.	213.19	"	5	7	7
6.	594	"	3	5	19	21.	326	"	4	7	19
7.	273	"	5	4	23	22.	487	"	3	5	26
8.	416	"	8	7	14	23.	216	"	4	4	11
9.	325.34	"	2	10	10	24.	342.26	"	6	8	10
10.	437	"	3	0	16	25.	319.35	"	4	7	5
11.	286.75	"	1	6	15	26.	417.26	"	5	8	5
12.	113.25	"	4	2	8	27.	318.34	"	9	7	2
13.	496	"	5	3	17	28.	917.26	"	4	8	26
14.	384	"	6	5	6	29.	213.25	"	5	6	4
15.	316.25	"	4	9	4	30.	518.40	"	2	7	25

119. To Find the Interest at 6 Per Cent on \$100, \$25, \$99, etc.

We wish to find the interest at 6% on \$100 for 4 yr. 9 mo. 17 da.

What is the interest on \$1 for 4 yr. 9 mo. 17 da.?

Change the fraction ⅚ to a decimal. What result do you obtain?

How shall we multiply \$.287833 by 100?

What, then, is the interest on \$100 for 4 yr. 9 mo. 17 da.?

How, evidently, can we find the interest at 6% for any time

On \$10?	On \$10,000?	On \$100,000?
On \$1000?	On \$1,000,000?	

Give a special rule for finding the interest at 6% for any time when the number of dollars in the principal is any power of 10.

* *

How, evidently, can we find the interest on any principal when the number of dollars is some convenient part of a power of 10? how when the number of dollars is a little more or less than some power of 10?

How, then, can we most conveniently find the interest

On \$25?	On \$125?	On \$2500?
On \$99?	On \$998?	On \$100.50?

Ex. 1. Find the interest at 6% on \$1000 for 5 yr. 7 mo. 28 da.

Ex. 2. Find the interest at 6% on \$2.50 for 6 yr. 8 mo. 2 da.

Ex. 3. Find the interest at 6% on \$99.75 for 3 yr. 11 mo 13 da.

Ex. 4. Find the interest at 6% on \$101 for 4 yr. 9 mo. 23 da.

Ex. 5. Find the interest at 6% on \$500 for 3 yr. 7 mo. 20 da.

Ex. 6. Find the interest at 6% on \$5400 for 5 yr. 9 mo. 7 da.

SOLUTIONS.

(1)	(2)
\$.339·666	\$.4·0033 \$1.

(3)	(4)
\$.23·7166 .0593 \$23.657	\$.28·8833 ·2888 \$29.17

(5)	(6)
\$.218·333 \$109.17	\$.34·61⅙ 207.70 \$1869.30

EXPLANATIONS.

Ex. 1. The interest on \$1 for 5 yr. 7 mo. 28 da. is \$.339⅔, or \$.339666, and the interest on \$1000 is 1000 times \$.339666, or \$339.67.

Ex. 2. The interest on \$1 for 6 yr. 8 mo. 2 da. is \$.40033; on \$10, 10 times \$.40033, or \$4,0033; and on \$2.50, ¼ of 4.0033, or \$1.

Ex. 3. The given principal, \$99.75, equals \$100 less ¼ of a dollar. The interest on \$100 for the given time is \$23.7166, and on ¼ of a dollar ¼ of \$.237166, or \$.0593. The required interest, therefore, is \$23.7166 less \$.0593, or \$23.66.

Ex. 4. The interest on \$101 for 4 yr. 9 mo. 23 da. is \$28.8833, the interest on \$100, plus \$.2888, the interest on \$1, or \$29.17.

Ex. 5. We first find the interest on \$1000 for 3 yr. 7 mo. 20 da. This interest is \$218.333, and the interest on \$500 is ½ of \$218.333, or \$109.17.

Ex. 6. The interest on \$100 for 5 yr. 9 mo. 7 da. is \$34.61⅙, and on \$5400, 54, or 9 times 6 times, \$34.61⅙, or \$1869.30.

Note. Observe that in Ex. 6 we were able to express the interest on 100 dollars in such a form that the fraction it contained was a fraction of a cent and disappeared on being multiplied by the first of the two remaining factors.

Remember that an error in the interest on one dollar is increased in the total interest as many fold as there are dollars in the principal, and reject a fraction from the interest on one dollar only when the principal is of one of the classes treated in this article.

Observe that every interest is the product of three elements, a principal, a time, and a rate. Observe also that the number of units in any two of these elements can be interchanged. Thus, the interest on \$47 for 25 days equals the interest on \$25 for 47 days.

Ex. 125.

Find the interest at 6 per cent on

			YR.	MO.	DA.				YR.	MO.	DA.
1.	\$100	for	4	7	16	15.	\$480	for	1	5	22
2.	25	"	5	6	21	16.	5000	"	5	6	7
3.	10	"	3	5	17	17.	10	"	4	8	2
4.	99	"	4	9	16	18.	99	"	3	7	13
5.	1000	"	3	6	28	19.	12.50	"	4	5	1
6.	250	"	5	8	16	20.	102	"	6	7	4
7.	2.50	"	4	9	13	21.	100	"	5	8	13
8.	101	"	5	5	16	22.	75	"	4	9	23
9.	250	"	4	7	29	23.	37.50	"	5	6	25
10.	10000	"	4	6	9	24.	62.50	"	2	3	10
11.	999	"	3	7	16	25.	87.50	"	5	9	23
12.	500	"	4	3	5	26.	375	"	4	7	28
13.	700	"	5	9	8	27.	625	"	5	2	16
14.	2500	"	4	8	2	28.	875	"	4	9	9

120. To Find the Interest at Any Per Cent for Years and Months.

We wish to find the interest on $80 for 2 yr. 6 mo. at 5 per cent.

What is the interest on $80 for 1 yr. at 5 per cent?

Six months are what part of a year?

By what, then, must we multiply the $4 to get the total interest?

What result do we thus obtain?

How, evidently, can we obtain the interest on a given principal for a given number of years and months at any per cent?

We wish to find the interest on $45 at 8 per cent for 3 yr. 9 mo.

9 months are what part of a year?

What is the interest on $1 at 8 per cent for 3 yr.? for 9 mo.? for 3 yr. 9 mo.

.30 times $45 equals what?

What, then, is the required interest?

By what second process, then, can we find the interest on a given principal for a given number of years and months at any per cent?

Ex. 1. Find the interest at 4½ per cent on $324.76 for 2 yr. 8 mo.

Ex. 2. Find the interest at 10 per cent on $494.36 for 5 yr. 10 mo.

Ex. 3. Find the interest at 7 per cent on $427.38 for 4 yr. 7 mo.

Ex. 4. Find the interest at 5 per cent on $185.36 for 3 yr. 8 mo.

SOLUTIONS.

(1)
$38.97

(2)
$49·4.36
247.18
41.197
$288.38

(3)
$ 29.9166
119.6664
14.9583
2.4931
$137.12

(4)
$37.072
3.089
$33.983

EXPLANATIONS.

Ex. 1. The interest on $1 for 2 years is twice $.04½, or $.09.

Two years are 24 months. Eight months are ⅓ of 24 months. The interest, therefore, on $1 for 8 months is ⅓ of $.09, or $.03, and for 2 yr. 8 mo. is $.09 plus $.03, or $.12. .12 times 324.76 equals 38.97. The required interest, therefore, is $38.97.

Ex. 2. The interest on $494.36 for 1 year is 10 per cent of 494.36, or $49.436. The interest for 5 yr. 10 mo., therefore, is 5⅚ times $49.436, or $247.18 plus ⅙ of $24718, or $288.38.

Ex. 3. The interest for 1 year is .07 of $427.38, or $29.9166, and for 4 years, 4 times $29.9166, or $119.6664.

Give a fourth rule for finding the interest on any principal for any time at any rate.

When will it be advisable to apply this rule?

* *

We wish to discover a direct method of computing interest on any principal for any time at 12 per cent.

What is the interest at 12 per cent on $1 for 1 year?

How will the time required at 12 per cent for $1 to gain 1 cent compare with the time at 6 per cent?

How many months, then, will be required at 12 per cent for $1 to gain 1 cent?

How many days, evidently, will be required to gain 1 mill?

Give a rule for writing out the interest at 12 per cent on $1 for any number of years, months, and days.

Ex. 1. Find the interest at $8\frac{1}{4}$ per cent on $100 for 3 yr. 7 mo. 26 da.

Ex. 2. Find the interest at 11 per cent on $60 for 2 yr. 6 mo. 23 da.

Ex. 3. Find the interest at 4 per cent on $216.75 for 3 yr. 9 mo. 5 da.

Ex. 4. Find the interest on $1200 for 3 yr. 7 mo. 23 da. at 5 per cent.

Ex. 5. Find the interest on $100 at 9 per cent for 2 yr. 8 mo. 12 da.

SOLUTIONS.

(1)	(2)	(3)
$21·933	$9.23	$72.25
7.311	101.53	$.45\frac{1}{6}$
.914	$16.92	·1204
$30.16		3 6125
	(5)	28 900
(4)	$.24·3^0$	$32.63
$.218·833		

EXPLANATIONS.

Ex. 1. The interest on the given principal for the given time at 6 per cent is $21.933.

$8\frac{1}{4}$ per cent equals 6 per cent plus 2 per cent plus $\frac{1}{4}$ per cent. The interest at 2 per cent is $\frac{1}{3}$ of the interest at 6 per cent, or $7.311, and the interest at $\frac{1}{4}$ per cent is $\frac{1}{8}$ of the interest at 2 per cent, or $.912. The total interest, therefore, is the sum of the three interests, or $30.16.

Ex. 2. 2 yr. 6 mo. 23 da. equal (30 times 30) plus 23, or 923, days. The interest on $60 at 6 per cent for 923 days equals, evidently, the interest on $923 for 60 days, or $9.23.

To find the interest at 11 per cent, we multiply the interest at 6 per cent by 11 and divide the product by 6. The result thus obtained is $16.92.

Ex. 3. The interest on \$216.75 for a given time at 4 per cent is evidently the same as the interest on $\frac{1}{3}$ of \$216.75, or \$72.25, for 3 times 4, or 12, per cent.

The interest on \$1 at 12 per cent for the given time is 36 plus 9 plus $\frac{1}{6}$, or $45\frac{1}{6}$, cents. The given interest, therefore, is $.45\frac{1}{6}$ times \$72.25, or \$32.63.

Ex, 4. We observe that the given principal is $\frac{6}{5}$ of the convenient principal, \$1000, and that the given rate is $\frac{5}{6}$ of the convenient rate, 6 per cent. We therefore find the required interest by finding the interest for the given time on \$1000 at 6 per cent. This interest is \$218.83.

Ex. 5. The interest on \$1 at 9 per cent for the given time is $\frac{3}{2}$ of \$.162, or \$.243. The required interest, therefore, is 100 times \$.243, or \$24.30.

Ex. 127.

[Before solving each exercise think of the different possible methods of solution, and use the simplest.]

Find the interest on

1. \$234.60 at 9 per cent for 3 yr. 5 mo. 16 da.
2. \$500 at 8 per cent for 4 yr. 11 mo. 14 da.
3. \$784.36 at 9 per cent for 3 yr. 5 mo. 16 da.
4. \$927 at 10 per cent for 2 yr. 6 mo. 30 da.
5. \$600 at $8\frac{1}{2}$ per cent for 3 yr. 5 mo. 10 da.
6. \$995 at $7\frac{1}{4}$ per cent for 2 yr. 3 mo. 21 da.
7. \$376.39 at 9 per cent for 6 yr. 4 mo. 14 da.
8. \$2347.50 at 11 per cent for 1 yr. 0 mo. 10 da.
9. \$320 at 7 per cent for 2 yr. 8 mo. 7 da.
10. \$875 at $4\frac{1}{2}$ per cent for 4 yr. 3 mo. 15 da.
11. \$99.50 at 3 per cent for 1 yr. 4 mo. 20 da.
12. \$398 at $5\frac{1}{4}$ per cent for 2 yr. 3 mo. 21 da.
13. \$476 at 12 per cent for 6 yr. 7 mo. 28 da.
14. \$50 at 4 per cent for 3 yr. 4 mo. 12 da.
15. \$310 at $4\frac{1}{2}$ per cent for 1 yr. 9 mo. 29 da.
16. \$50 at $9\frac{1}{2}$ per cent for 2 yr. 6 mo. 10 da.
17. \$250 at 10 per cent for 3 yr. 3 mo. 5 da.
18. \$30 at 12 per cent for 2 yr. 4 mo. 12 da.
19. \$6 at $7\frac{1}{2}$ per cent for 3 yr. 3 mo. 17 da.
20. \$183.75 at 3 per cent for 2 yr. 2 mo. 20 da.
21. \$750 at 5 per cent for 2 yr. 5 mo. 19 da.
22. \$1650 at 7 per cent for 3 yr, 7 mo. 24 da.

122. To Find the Interest at Any Rate for Any Number of Days.

How many months are required for $1 to gain 5 cents of interest at 5 per cent? how many days?

How many days, then, are required for $1 to gain 1 cent of interest?

How, then, can we find the interest on any principal at 5 per cent for 72 days?

How can we find the interest

For 36 days?	For 9 days?	For 60 days?
For 24 days?	For 8 days?	For 81 days?
For 12 days?	For 6 days?	For 108 days?

Give a general rule for for finding the interest at 5 per cent on any principal for any number of days.

* *

Form by similiar reasoning a rule for finding the interest on any principal for any number of days

At 5 per cent.	At 4 per cent.	At 4½ per cent.
At 8 per cent.	At 9 per cent.	At 3 per cent.

Apply each of these rules to an exercise of your own construction.

Ex. 1. Find the interest at 5 per cent on $325 for 48 days.

Ex. 2. Find the interest at 9 per cent on $216.34 for 25 days.

SOLUTIONS.

(1)	(2)	(3)
$3.25	$1.0817	$8·94.75
1.08	.2704	2.98 25
$2.17	$1.35	$11.93

Ex. 3. Find the interest at 12 per cent on $894.75 for 40 days.

EXPLANATIONS.

Ex. 1. A dollar at 5 per cent gains 1 cent of interest in $\frac{1}{5}$ of 360, or 72, days. The required interest, therefore, is .01 of $325, or $3.25, the interest for 72 days, less ⅓ of $3.25, or $1.08, the interest for 24 days, or $2.17.

Ex. 2. A dollar at 9 per cent gains 1 cent of interest in $\frac{1}{9}$ of 360, or 40, days. The required interest, therefore, is ½ of .01 of $216.34, or $1.0817, the interest for 20 days, plus ¼ of $1.0817, or $.2704, the interest for 5 days, or $1.35.

Ex. 3. A dollar at 12 per cent gains 1 cent of interest in $\frac{1}{12}$ of 360, or 30, days. The required interest, therefore is .01 of $894.75, or $8.9475, the interest for 30 days, plus ⅓ of $8.9475, or $2.9825, the interest for 10 days, or $11,93.

Ex. 128.

Find the interest on

1. $843.75 for 53 days at 8 per cent.
2. $275 for 39 days at 5 per cent.
3. $318.26 for 33 days at 11 per cent.
4. $150 for 28 days at 4 per cent.
5. $324.50 for 63 days at 7 per cent.
6. $900 for 43 days at 10 per cent.
7. $728.16 for 17 days at 3 per cent.
8. $450 for 21 days at 9 per cent.
9. $216.24 for 84 days at 2 per cent.
10. $356.24 for 73 days at 8 per cent.

Ex. 129.

Find the time between the dates in each of the following exercises according to the principles presented in Article 104.

Compute the interest in the first 10 exercises at 6 per cent.

1. What is the interest on $835.75 from June 25, 1898 to May 10, 1902?

2. What is the interest on $4780 from Jan. 5, 1894 to March 24, 1897?

3. What is the interest on $865.40 from July 22, 1901 to March 4, 1903?

4. What is the interest on $250 from Nov. 10, 1897 to July 5, 1901?

5. What is the interest on $97 from Oct. 3, 1900 to March 10, 1902?

6. What is the interest on $75 from June 23, 1891 to Aug. 4, 1894?

7. What is the interest on $5000 from May 9, 1901 to June 13, 1905?

7. What is the interest on $875.50 from April 2, 1897 to June 12, 1901?

8. What is the interest on $2600 from Jnly 11, 1898 to May 6, 1902?

10. What is the interest on $4800 from April 7, 1903 to June 9, 1906.

Ex. 130.

1. Find the interest at 4 per cent on $250 from Apr. 18, 1902 to Feb. 6, 1904.

2. Find the interest at 7 per cent on $373.47 from Dec. 17, 1898 to Apr. 10, 1903.

3. Find the interest at 8 per cent on $100 from Apr. 7, 1901 to Dec. 10, 1904.

4. Find the interest at 3 per cent on $56 from Feb. 1, 1899 to Feb. 16, 1902.

5. Find the interest at 5 per cent on $341.26 from June 10, 1900 to Aug. 30, 1905.

6. Find the interest at 3 per cent on $333 from May 7, 1898 to Apr. 4, 1903.

7. Find the interest at 6 per cent on $42.35 from Aug. 1, 1901, to Nov. 4, 1905.

8. Find the interest at 5 per cent on $25 from Dec. 16, 1900 to Nov. 9, 1904.

9. Find the interest at 9 per cent on $32.88 from Sept. 1, 1898 to Feb. 24, 1901.

10. Find the interest at 2 per cent on $1000 from Feb. 11, 1898 to March 10, 1902.

11. Find the interest at 7 per cent on $497.56 from Aug. 24, 1899 to. Apr. 5, 1901.

13. Find the interest at 4½ per cent on $750 from Dec. 24, 1901 to Jan. 8, 1904.

14. Find the interest at 7¼ per cent on $216.29 from June 13, 1900 to Sept. 25, 1903.

15. Find the interest at 8 per cent on $1250 from Oct. 11, 1903 to Apr. 3, 1905.

16. Find the interest on $60 at 6 per cent from May 24, 1902 to July 7, 1902.

17. Find the interest at 5 per cent on $25000 from May 17, 1899 to March 12, 1902.

18. Find the interest at 3½ per cent on $325.98 from Nov. 16, 1897 to June 3, 1901.

19. Find the interest at 5 per cent on $650 from Jan. 7, 1903 to April 5, 1905.

123. Annual Interest.

A man gives a note for $1000, agreeing to pay interest upon it at 6 per cent at the end of each year. When, however, at the end of three years and six months, he pays the note, he has paid no interest upon it. In determining the amount due, the holder of the note claims besides the principal and interest recompense for the non-payment of the several interests at the dates at which they became due.

When ought the first year's interest to have been paid?
When was it paid?
For how long a time, then, did the holder of the note lose the use of the first year's interest?
How should he be compensated for this loss?

When should the second year's interest have been paid?
When was it paid?
For how long a time, then, did the holder of the note lose the use of the second year's interest?
How should he be compensated for this loss?

When should the third year's interest have been paid?
When was it paid?
For how long a time, then, did the holder of the note lose the use of the third year's interest?
How should he be compensated for this loss?

* *

For how long a time have we found that the holder of the note should receive interest

On the first year's interest?
On the second year's interest?
On the third year's interest?

The interest on $60 for 2 yr. 6 mo. plus the interest on $60 for 1 yr. 6 mo. plus the interest on $60 for 6 months equals the interest on $60 for how many years and months?
By what short process, then, can we find the total amount that should be paid as compensation for the non-payment of the several interests at the dates at which they becane due?

* *

That process of computing interest which we have followed in the preceding problem is called **Annual Interest.** That part of the interest computed directly on the $1000 may be referred to as **Direct Interest,** and that computed directly on the $60, and indirectly on the $1000, as **Indirect Interest.** Hence the following definitions:

Annual Interest is that process of computing interest by

which interest not paid when it becomes due is subject to simple interest from the time that it becomes due to the time of settlement.

Direct Interest is that part of the annual interest computed directly on the principal.

Indirect Interest is that part of the annual interest which is paid as compensation for each year's interest's not being paid when it becomes due, and which, therefore, is computed indirectly on the principal.

The process of computing annual interest is made evident by the preceding definitions. But fix thoroughly the following facts:

Annual interest is made up of direct interest and indirect interest.

In computing direct interest, the principal is the face of the note and the time is the time of the note.

In computing indirect interest, the principal is a year's interest on the face of the note and the time is the total time that the several indirect interests remain unpaid.

124. To Find the Time for the Indirect Interest.

A note bearing annual interest is paid at the end of 5 years 2 months 5 days. We wish to find the time for the indirect interest.

4	2	5
3	3	5
2	2	5
1	2	5
	2	5

For how long a time has the first year's interest remained unpaid?

The second year's interest? The third year's interest? The fourth year's interest? The fifth year's interest?

How, then, may we find the total time for the indirect interest?

In finding the time for the indirect interest how many times do we use the 5 days as an addend?

How many times the 2 months?

How does the number of times in each case compare with the number of years in the period of the note?

By what short process, then, might we have found the number of months and days for the indirect interest?

Explain the reason for the relation we have discovered.

Give a general rule for finding the number of months and days for an indirect iuterest.

Give a rule for finding the number of years for an indirect interest.

Ex. 1. A note bearing annual interest is paid at the end of 4 yr. 8 mo. 18 da. What is the time for the indirect interest?

SOLUTIONS.

(1)			(2)		
$\overset{2}{4}$	$\overset{2}{8}$	18	$\overset{4}{5}$	$\overset{4}{11}$	28
8	10	12	14	11	20

Ex. 2. A note bearing annual interest is paid at the end of 5 yr. 11 mo. 28 da. What is the time for the indirect interest?

Explanations.

Ex. 1. If we should write out the several periods that the interest due at the end of each year remains unpaid, the months and the days would each appear as many times as there are units in the number of years. We therefore find the number of months and days in the time for the indirect interest by multiplying the given number of months and days by 4, the given number of years.

4 times 8 mo. 18 da. are 2 yr. 10 mo. 12 da. Combining this result with 3 plus 2 plus 1, or 6, years, we find the total time for the indirect interest to be 8 yr. 10 mo. 12 da.

Ex. 2. Explain the solution of this exercise.

NOTE 1. Observe that 18 days may be thought of as ½ month plus 3 days, and 28 days as 1 month less 2 days. 4 times 18 days, therefore, will be 4 half-months, or 2 months, plus 12 days; and 5 times 28 days will be 5 months less 10 days, or 4 months plus 20 days.

Use artifices like these at least to test the accuracy of your results by the ordinary methods.

NOTE 2. The years for the indirect interest may be found by multiplying the given number of years by 1 less than the number, and dividing by 2. Thus if the number of years for the direct interest is 5, the number for the indirect interest will be (5 times 4) divided by 2, or 5 times 2, or 10.

Ex. 131.

Find the time for the indirect interest when the time of the note is

1. 3 years.	9. 2 yr. 3 mo.	17. 1 yr. 3 mo. 7 da.
2. 4 years.	10. 5 yr. 1 mo.	18. 2 yr. 5 mo. 8 da.
3. 5 years.	11. 4 yr. 3 mo.	19. 4 yr. 3 mo. 12 da.
4. 2 years.	12. 5 yr. 5 mo.	20. 5 yr. 7 mo. 16 da.
5. 1 year.	13. 4 yr. 7 mo.	21. 4 yr. 3 mo. 26 da.
6. 7 years.	14. 2 yr. 11 mo.	22. 4 yr. 7 mo. 15 da.
7. 9 years.	15. 3 yr. 8 mo.	23. 3 yr. 5 mo. 16 da.
8. 6 years.	16. 3 yr. 5 mo.	24. 2 yr. 10 mo. 14 da.

Ex. 1. Find the annual interest on $1000 for 3 yr. 7 mo. 23 da. at 6 per cent.

Ex. 2. Find the annual interest on $324.75 for 5 years at 7 per cent.

Ex. 3. Find the annual interest on $227.34 for 5 yr. 6 mo. 16 da. at 10 per cent.

SOLUTIONS.

(1)

$\$1000$ for	$\overset{1}{3}$	$\overset{2}{7}$	23	=	$218.83
$60 "	4	11	9	=	17.79
.2·965					$236.62

(2)

$324.75	for	5 yr.	=	$113.66
$2·2.7325	"	10 "	=	15.91
				$129.57

(3)

$\$227.34$	for	$\overset{2}{5}$	$\overset{2}{6}$	16	=	$126.05
$22.73		12	8	20	=	28.92
$75.78			76⅓			$154.97
.1516			758			
$75.6·284		1·3638				
		15 911				
		17.3·506				

EXPLANATIONS.

Ex. 1. We first find the principal and the time for the indirect interest. The principal is .06 of $1000, or $60. The time is 3 times 7 mo. 23 da, plus (2 plus 1) years, or 4 yr. 11 mo. 9 da. The remainder of the solution is simply the solving of two exercises in simple interest. Solving these, we find the direct interest to be $218.83, the indirect interest to be $17.79, and the entire annual interest to be the sum of these two interests, or $236.62.

Ex. 2. Explain the solution of this exercise.

Ex. 3. Explain the solution of this exercise.

NOTE. Observe the following points in the preceding exercises. The orders below cents in a year's interest on the principal are rejected when the interest is used as the principal for the indirect interest, but are retained in finding the direct interest.

Thus, in Ex. 2. $22.7325 are multiplied by 5 to find the direct interest, but $22.73 are used as the principal in finding the indirect interest.

Ex. 132.

Find the annual interest at 6 per cent on

1. $140 for 9 yr.
2. $75.30 for 2 yr.
3. $500 for 3 yr. 6 mo.
4. $287 for 2 yr. 5 mo.
5. $25 for 5 yr. 10 mo.
6. $110 for 1 yr. 1 mo.
7. $333 for 4 yr. 7 mo.
8. $230 for 4 yr. 1 mo.
9. $87.50 for 4 yr. 7 mo. 18 da.
10. $450 for 2 yr. 5 mo. 24 da.
11. $54.60 for 3 yr. 7 mo. 13 da.
12. $100 for 5 yr. 5 mo. 7 da.
13. $50 for 3 yr. 8 mo. 12 da.
14. $45.50 for 2 yr. 5 mo. 29 da.
15. $250 for 2 yr. 2 mo. 2 da.
16. $100 for 4 yr. 4 mo. 6 da.

Ex. 133.

Find the annual interest on

1. $100 for 4 years at 6 per cent.
2. $25 for 2 yr. 9 mo. at 5 per cent.
3. $346.72 for 5 yr. 8 mo. 17 da. at 4½ per cent.
4. $2000 for 7 yr. 3 mo. 10 da. at 8 per cent.
5. $765.25 for 9 yr. 4 mo. 5 da. at 10 per cent.
6. $200 for 3 yr. 7 mo. 24 da. at 3 per cent.
7. $75 for 1 yr. 7 mo. 19 da. at 9 per cent.
8. $1000 for 3 yr. 8 mo. 27 da. at 12 per cent.
9. $48 for 2 yr. 5 mo. 18 da. at 11 per cent.
10. $1250 for 4 yr. 6 mo. 18 da. at 7½ per cent.

Ex. 134.

Find the annual interest on

1. $275 from March 24, 1898 to Dec. 8, 1901 at 5 per cent.
2. $25000 from March 3, 1900 to Feb. 5, 1905 at 7½ per cent.
3. $896.24 from Sept. 27, 1901 to May 20, 1903 at 3 per cent.
4. $300 from Oct. 6, 1902 to Dec. 15, 1905 at 4½ per cent.
5. $39.76 from Dec. 13, 1900 to July 3, 1905 at 8 per cent.
6. $484.37 from Nov. 15, 1899 to Feb. 7, 1904 at 5 per cent.
7. $5600 from May 3, 1897 to Dec. 15, 1900 at 4 per cent.
8. $25 from Oct. 8, 1901 to Sept. 3, 1905 at 9 per cent.
9. $350 from Sept. 3, 1903 to Dec. 3, 1906 at 3 per cent.
10. $284 from Nov. 28, 1899 to Dec. 7, 1903 at 7 per cent.

Ex. 135.

Find the annual interest on

1. $256.35 from Jan. 19, 1891 to Feb. 1, 1895 at 9 per cent.
2. $100 from Dec. 18, 1897 to Feb. 6, 1900 at 5 per cent.
3. $4560 from Sept. 17, 1902 to March 2, 1906 at 8 per cent.
4. $228.56 from Oct. 8, 1893 to June 7, 1898 at 3 per cent.
5. $285.70 from Nov. 5, 1895 to Aug. 29, 1897 at 4 per cent.
6. $99.98 from Aug. 5, 1903 to July 24, 1905 at 4¼ per cent.
7. $8000 from Sept. 23, 1898 to May 6, 1902 at 6 per cent.
8. $725.45 from Nov. 30, 1903 to April 8, 1906 at 7 per cent.
9. $45.37 from Oct. 26, 1897 to July 3, 1900 at 9 per cent.
10. $26.43 from March 4, 1897 to July 7, 1902 at 5 per cent.
11. $825.50 from May 2, 1901 to Oct. 25, 1903 at 4 per cent.
12. $235.75 from Aug. 6, 1899 to Nov. 9, 1902 at 3½ per cent.
13. $247 from Aug. 24, 1902 to Dec. 7, 1905 at 6 per cent.

125. Compound Interest.

That process of computing interest by which all interest that is not paid when it becomes due becomes a part of the principal is called **Compound Interest.** Compound interest is authorized by law in but few, if any, States.

Compound Interest.

The principles of compound interest are self-evident. Their applications will be made clear by the following solution.

What is the amount at compound interest at 5 per cent, payable semi-annually, of $236.75 for 4 yr. 7 mo. 6 da.?

SOLUTION.

$236.75	
5.92	
242.67	1
6.07	
248.74	2
6.22	
254.96	3
6.37	
261.33	4
6.53	
267.86	5
6.70	
274.56	6
6.86	
281.42	7
7.04	
288.46	8
7.21	
295.67	9
1.48	
297.15	

EXPLANATION.

The compound interest on the given principal for 4 yr. 6 mo. at 5 per cent payable semi-annually is equivalent to the interest for 9 periods at $2\frac{1}{2}$ per cent.

To find the amount due at the end of each period, we increase the amount due at the beginning by .025, or $\frac{1}{40}$, of itself. We thus find the amount at the end of 4 yr. 6 mo. to be $295.67.

We next find the amount of $295.67 for 1 mo. 6 da. at 5 per cent. This amouut is $297.15. The required amount, therefore, for the given time at the given rate is $297.15.

Ex. 136.

1. Find the compound interest at 6 per cent on $294 for 4 yr. 7 mo. 16 da.

2. What is the amount of $400 at 5 per cent compound interest for 3 yr. 9 mo. 24 da.?

3. What is the compound interest, payable semi-annually, on $324.87 at 5 per cent for 3 yr. 9 mo. 24 da?

4. What is the compound interest, payable quarterly, at 8 per cent on $1000 from May 1, 1901 to July 9, 1905?

5. Find the simple and the compound interest at 6 per cent on $870.36 from April 24, 1901 to July 22, 1904.

6. Find the simple, the annual, and the compound interest at 7 per cent on $500 from Oct. 4, 1902 to March 29, 1906.

7. Find the compound interest at 8 per cent from May 23, 1901 to March 16, 1905 if compounded semi-annually; if compounded annually; if compounded quarterly.

126. To Apply a Table of Compound Interest.

Explain the process of finding from the table on the following page 7 per cent of any principal for 20 years; for 20 plus 15, or 35, years.

Give a rule for finding from a table the amount at compound interest on any principal at any rate for any time when the number of years is not greater than the number given in the table; when it is greater.

Find the amount at compound interest at 6 per cent of $10 from the landing of Columbus to Oct. 12, 1892.

SOLUTION.

3 20	1
2 56	
10 24	2
8 19	
32 76	3
26 21	
104 84	4
83 87	
335 48	5
268 38	
1073 52	6
858 82	
3435 28	7
2748 22	
10992 88	8
8794 30	
35177 20	9
28141 76	
112567 04	10
90053 63	
360214 52	11
288171 62	
1152686 48	12
922149 18	
3688596 72	13
2950877 38	
11803509 52	14
9442807 62	
37771230 48	15
30216984 38	
120867937 52	16
96694350 02	
386777400 08	17
309421920 06	
1237687680 24	18
990150144 19	
3960600576 76	19
3168480461 41	
12673921845·6·4	20

EXPLANATIONS.

The time between the two dates is 400 years, or 20 periods of 20 years each. The amount of $1 at 6 per cent for 20 years, extended to two decimal places, is $3.20, or 3.2 dollars. We therefore find the amount for the given time by multiplying the amount at the beginning of each period by 3.2.

Performing these multiplications, and rejecting all decimal orders below hundredths, we find the amount of $1 for the given time at the given rate to be $12,673,921,845,64. The required amount, therefore, is 10 times $12,673,921,845.64, or $126,739,218,456.40.

NOTE. The immense cumulative power of compouud interest is well illustrated by the preceding problem, the result showing that a debt of $10 contracted in 1492 could not now be paid, if allowed to draw compound interest, from all the wealth possessed at the present time by the entire New World that Columbus discovered. In recognition of this fact, apparently, Vermont and New Hampshire and certain other States, either by special statutes or through decisions of their courts, in the case of notes bearing interest annually have laid down the law that the yearly interests accruing on such notes shall not be compounded with the principal, but that instead they shall draw simple interest from the time they become due to the time of settlement. The application of this principle has already been explained under Annual Interest.

COMPOUND INTEREST TABLE SHOWING THE AMOUNT OF $1, FROM 1 TO 20 YEARS, AT 3, 4, 5, 6, 7, AND 8 PER CENT.

Years.	3 per cent.	4 per cent.	5 per cent.	6 per cent.	7 per cent.	8 per cent.
1	1.0300	1.0400	1.0500	1.0600	1.0700	1.0800
2	1.0609	1.0816	1.1025	1.1236	1.1449	1.1664
3	1.0927	1.1249	1.1576	1.1910	1.2250	1.2597
4	1.1255	1.1699	1.2155	1.2625	1.3108	1.3605
5	1.1593	1.2167	1.2763	1.3382	1.4026	1.4693
6	1.1941	1.2653	1.3401	1.4185	1.5007	1.5869
7	1.2299	1.3159	1.4071	1.5036	1.6058	1.7138
8	1.2668	1.3686	1.4775	1.5938	1.7182	1.8509
9	1.3048	1.4233	1.5513	1.6895	1.8385	1.9990
10	1.3439	1.4802	1.6289	1.7908	1.9672	2.1589
11	1.3842	1.5395	1.7103	1.8983	2.1049	2.3316
12	1.4258	1.6010	1.7959	2.0122	2.2522	2.5182
13	1.4685	1.6651	1.8856	2.1329	2 4098	2.7196
14	1.5126	1.7317	1.9799	2.2609	2.5785	2.9372
15	1.5580	1.8009	2.0789	2.3966	2.7590	3.1722
16	1.6047	1.8730	2.1829	2.5404	2.9522	3.4259
17	1.6528	1.9479	2.2920	2.6928	3.1588	3.7000
18	1.7024	2.0258	2.4066	2.8543	3.3799	3.9960
19	1.7535	2.1068	2.5270	3.0256	3.6165	4.3157
20	1.8061	2.1911	2.6533	3.2071	3.8697	4.6610

Ex. 137.

Extend products in the following exercises to four decimal places.

1. What would have been the amount at 8 per cent compound interest of $1 from the landing of Columbus to the settlement at St. Augustine.

2. Find the amount at 7 per cent compound interest of $100 from the settlement of St. Augustine to the landing of the Pilgrims.

3. Find the amount at 6 per cent compound interest of $1000 from the landing of the Pilgrims to the Declaration of Independence.

4. Find the amount at 8 per cent compound interest of $100 from the Declaration of Independence to the close of the Rebellion.

5. What is the ratio of the compound interest on $1 for 100 years at 6 per cent to the compound interest at 3 per cent?

6. What is the ratio of the compound interest on $1 at 6 per cent for 50 years to the compound interest for 5 years?

7. What is the ratio of the compound interest on $1 for 28 years at 8 per cent to the compound interest for 7 years at 4 per cent?

127. To Find the Principal when the Interest, Time, and Rate are Given.

Three factors are to be multiplied together. Two of the factors are used, but a new factor is substituted for the third factor. The product thus obtained is 15 times the product that would have been obtained had all three of the given factors been used.

What, evidently, must be the ratio of the new factor to the third factor?

* *

We wish to find the principal which at a given per cent will gain a given interest in a given time.

We assume that $1 is the required principal and find the interest upon it for the given time and rate. The interest thus obtained is too small, the ratio of the given interest to it being 134.68.

What, evidently, must be the ratio of the required principal to the assumed principal?

What, then, is the assumed principal?

Give a rule for finding the principal when the time, the rate, and the interest are given.

SOLUTION.

What principal will gain $194.37 interest in 4 yr. 5 mo. 17 da. at 4½ per cent?

$$\begin{array}{r} .267\frac{5}{6} \\ .066\frac{23}{24} \\ \hline .200\frac{7}{8} \\ 1,607 \end{array}$$

```
   967 62
1554.960
1446 3
  1086 6
   964 2
   1224 0
   1124 9
     991 0
     964 2
      26 8
```

EXPLANATION.

We assume that $1 at the given rate in a given time will gain the given interest. We find, however, that the interest on $1 is too small, the ratio in its simplest form of the given interest to it being $\frac{1554960}{1607}$. This ratio expressed as a mixed number is 967.62. The required interest, therefore, is 967.62 times $1, or $967.62.

(Explain the simplifying of the ratio of $194.37 to $.200⅞.)

NOTE. If the amount instead of the interest is given it is evident that the process will be the same. That is, $1 will be assumed to be the required principal, and the ratio will be found of the given amount to the amount of $1.

Ex. 138.

1. What principal at 6 per cent will produce $84.75 interest in 3 yr. 7 mo. 15 da.

2. What principal at 5 per cent will produce $324 interest in 5 yr. 7 mo. 24 da.

3. What principal at 7 per cent will amount to $1575.70 in 5 yr. 8 mo. 29 da.

128. To Find the Time when the Principal, Interest, and Rate Are Given.

We wish to find the time that will be required for a given principal at a given rate to gain a given interest.

We assume that 1 year is the required time and find the interest for this time upon the given principal at the given rate. The interest thus obtained is too small, the ratio of the correct interest to it being 3.93.

What, evidently, must be the ratio of the required time to the assumed time?

What, then, is the required time?

Give, then, a rule for finding the time when the principal, the interest, and the rate, are given.

In what time will $896.27 produce $216.34 interest at 7 per cent.

SOLUTION.

```
216·3400|_62·7389_
188 2167  3  5  11
 28 1233
337 4796
313 6945
 23 7851
713 5530
627 389
 86 1640
```

EXPLANATION.

We assume that the given principal at the given rate will produce the given interest in 1 year. We find, however, that 1 year is too short a time, the ratio of the given interest to it being $\frac{2163400}{627389}$. This ratio expressed as a mixed number is $3\frac{281233}{627389}$. The required time, therefore, is $3\frac{281233}{627389}$ years.

The fraction of a year we reduce to months by multiplying by 12 and find it to be $5\frac{237851}{627389}$ months. The fraction of a month we reduce to days and find it to be 11 days. The required time, therefore, is 3 yr. 5 mo. 11 da.

NOTE. Observe that the time will not necessarily be such a part of the year as can be reduced to an integral number of months and days. When this is the case, it arises from the fact that a decimal of one or more orders would be rejected in obtaining the given interest. The fraction, however, should not in any case be retained, but should be rejected or treated as an additional day as it is less or not less than ½.

Ex. 139.

1. In what time will $324 produce $95 interest at 7 per cent?

2. In what time will $500.27 produce $207.32 interest at 5 per cent?

1. In what time will $360.36 amount to $504 at 7 per cent?

4. In what time will $2034.65 produce $1269.90 interest at 8 per cent?

129. To Find the Rate when the Principal, Interest, and Time Are Given.

We wish to find the rate that is required for a given principal in a given time to gain a given interest.

We assume that 6 per cent is the required rate and find the interest at this rate upon the given principal for the given time. The interest thus obtained is too large, the ratio of the given interest to it being $\frac{2}{3}$.

What, evidently, must be the ratio of the required rate to the given rate?

What, then, is the required rate?

Give a rule for finding the rate when the principal, the interest, and the time are given.

Find the rate at which \$250 will produce \$108.62 in 4 yr. 9 mo. 28 da.

SOLUTION.

$$\begin{array}{r} \$.289.666 \\ 72.417 \\ 36.208 \\ \hline 108.62 \end{array}$$

EXPLANATION.

We assume that the given principal in the given time will produce the given interest at 6 per cent. We find, however, that 6 per cent is too low a rate, the interest at this rate being only \$72.417, instead of the given interest \$108.62.

We observe that the given interest is about $\frac{1}{2}$ larger than the interest at 6 per cent. Testing this relation, we find it is exactly $\frac{1}{2}$ larger. The required rate, therefore, is 6 plus 3, or 9, per cent.

NOTE. In finding the rate, if the result varies but an insignificant fraction from, for example, 5 per cent, $5\frac{1}{2}$ per cent, 7 per cent, 9 per cent, consider these answers the required result. If, however, there is a considerable variation, obtain the result by dividing the given interest by the interest at 1 per cent, and, if necessary, extend the division to thousandths.

Ex. 140.

1. At what rate will \$300 produce \$60 interest in 2 yr. 2 mo. 20 da.?

2. At what rate will \$124.36 produce \$41.66 interest in 4 yr. 5 mo. 18 da.?

3. At what rate will \$250 amount to \$390.49 in 5 yr. 7 mo. 13 da.?

4. At what rate will \$483.16 produce \$88. 44 cents interest in 3 yr. 7 mo. 28 da.?

5. At what rate will \$100 produce \$18.75 interest in 3 yr. 4 mo. 27 da,?

6. At what rate will \$1 double itself in 11 years?

130. Merchants' Rule for Partial Payments.

Frequently when debts have been incurred by the giving of a note or otherwise, payments are made upon them at irregular intervals. A knowledge of the principles governing the legal application of such payments, which are called Partial Payments, is essential to every one who has anything to do with the borrowing or loaning of money.

When a debt is paid a year or less from the date at which it was incurred, and partial payments have been made upon it, the process commonly employed is to find the amount of the debt from the date it was contracted to the date of settlement, and to subtract from this amount the amounts of the payments from the date at which each was made to the date of settlement. This method of computing partial payments is called the **Merchants' Rule.** Solve by this rule each of the following problems.

A full development of the terms "note," "indorsements," etc., will be found in the Article on Banking.

Ex. 141.

1. On a note for $500 dated March 15, 1900 and bearing 6 per cent interest the following indorsements are made:

June 10, 1900, $75; Aug. 13, 1900, $50; Nov. 20, 1900, $100; Dec. 18, 1900, $20; Feb. 8, 1901, $60; March 1, 1901, $70.

What is due on the note March, 15, 1901?

2. On a note for $875.50 dated Sept. 3, 1902 and bearing 4 per cent interest the following indorsements are made:

Nov. 5, 1902, $200; Dec. 25, 1902, $100; Apr. 4, 1903, $250; June 17, 1903, $100; Aug. 15, 1903, $75.

What is due on the note Sept. 3, 1903, the rate of interest being 7 per cent.

3. On a note for $60 dated July 1, 1902 and bearing 7 per cent interest the following indorsements are made:

Aug. 10, 1902. $10; Sept. 30, 1902, $5; Dec. 5, 1902, $8; Feb. 7, 1903, $4; March 19, 1903, $15; May 10, 1903, $7.

What is due on the note May 20, 1903?

4. On a note for $750 dated Sept. 24, 1903 and bearing 5 per cent interest the following indorsements are made:

Oct. 15, 1903, $125; Dec. 25, 1903, $200; Jan, 8, 1904, $25; March 6, 1904, $150; July 6, 1904, $100.

What is due on the note Sept. 24, 1904?

131. The United States Rule for Partial Payments.

When notes are made payable "with interest" and are not paid for more than a year, their amount is found, in case no rule has been established by the State in which the note was given, in accordance with a rule adopted by the Supreme Court of the United States called the **United States Rule.**

This rule seems to be based upon the theory (1) that the maker of the note shall in no case pay interest upon a larger principal than that upon which he contracted to pay, and (2) that payments shall not, directly or indirectly, bear any interest until all interest due has been paid. The fundamental principles of the rule, therefore, are

(1) That the amount of the principal, at simple interest, shall be found to the date at which the payment, or the sum of the payments, equals or exceeds the principal.

(2) That from this amount the payment, or sum of the payments, without interest, shall be subtracted.

A note bearing simple interest is given Sept. 6, 1900. Five dollars are paid upon it June 10, 1901; $.25 Dec. 5, 1902; and $60 Aug. 8, 1906. What, according to the U. S. Rule, is due upon the note Jan. 1, 1907?

SOLUTION.

Date.	Principal.	
Sept. 6, 1900.	$100	9 4
June 10, 1901.	$104.57	.04·566
	−5.00	5 1 28
After Payt.	99.57	99.57
	30.83	29871
Aug. 8, 1906.	130.40	30.8667
	−60.25	.033
After Payt.	70.15	
	1.67	4 23
Jan. 1, 1907.	**$71.82**	42090 1.68360

EXPLANATION.

The first payment is made June 10, 1901. The time to this date is only a little more than ¾ of a year, and the interest on the face of the note will be only a little more than ¾ of $6, or than $4.50. We therefore as the first step of the solution find the amount due on the note June 10, 1901. This amount is $100 plus $4.57, or $104.57. From it we subtract the given payment and thus obtain as the new principal on June 10, 1901 $104.57 less $5, or $99.57.

The second payment is made Dec. 5, 1902. The payment made on that date is, evidently, but a small part of the interest that has then accrued. We therefore pass over this payment to the third payment, which, evidently, is larger than the interest that has ac-

crued since the beginning of the second period. We therefore take the date at which it was made as the date at which we shall find the second new principal. The amount of our first new principal, $99.57 from June 10, 1901 to Aug. 8, 1906 is $99.57 plus $30.83, or $130.40. From this amount we subtract $.25 plus $60, or $60.25, the sum of the two payments. We thus obtain as the new principal from Aug. 8, 1906 $70.15.

The final step in the solution is to find the amount of $70.15 from Aug. 8, 1906 to Jan. 1, 1907, the date of settlement. This amount is $70.15 plus $1.67, or $71.82.

NOTE. 1. The United States Rule is favorable or unfavorable to the debtor according as he makes widely separated or frequent payments. Thus, a debtor paying $10 each 2 months on a debt of $1000 bearing 6 per cent interest is owing as much at the end of 5 years as a second debtor who does not pay a cent of interest until the end of the 5 years and then pays the same number of dollars as have been paid by the first. Observe, moreover, that the holder of the note, provided that he can loan under the same conditions all interest directly or indirectly arising from the debt, not only practically receives compound interest but is receiving interest compounded 6 times each year.

NOTE. 2. Suppose a note bearing simple interest to run through a long term of years, and suppose a payment equal to a year's interest to pe paid at the end of each year. Suppose that at the end of 30 years a settlement of the note is made according to the principles of the Merchants' Rule. By this settlement the holder of the note, unless he has given the subject careful study, will be unpleasantly suprised. Instead of receiving the original principal, as he might not unnaturally expect in view of the fact that he has received absolutely nothing except the interest upon it, he finds himself heavily in debt to the original debtor. The reason for this is not difficult to find. At the end of 17 years the payments of the debtor exceeded by (17 times 6) less 100, or 2, per cent the loan made by the creditor. After this year the interest accumulating in favor of the debtor exceeds, and by a constantly increasing quantity, that accumulating in favor of the creditor, and it will be only a question of time when this excess will become greater than the original debt and change the original relations of the two parties to it.

Ex. 142.

Solve by the United States Rule each of the problems under Ex. 143.

132. The Vermont and the New Hampshire Rules for Partial Payments.

In the computation of notes bearing annual interest on which partial payments have been made, Vermont and New Hampshire follow nearly the same method. The Vermont method is based upon a statute of the State, the New Hampshire method upon the decision of its Supreme Court. As will be seen, the principles of the two methods are exactly the same except in their treatment of payments.

COMMON PRINCIPLES.

1. **Each year shall be considered as beginning and ending with the day and month of the date of the note.**

2. **At the end of each year in which a payment is made there shall be computed**
   1. **The amount of principal.**
   2. **The amount of interest on principal.**
   3. **The amount of interest on interest.**

3. **The principal shall bear annual interest.**
   Interest on principal shall bear simple interest.
   Interest on interest shall bear no interest.

4. **Payments shall be applied**
   1. **To pay interest on interest.**
   2. **To pay interest on principal.**
   3. **To pay the principal.**

Up to this point the principles of the two rules have been identical. In their treatment of payments, they slightly differ. The Vermont Legislature assumed that the holder of the note, inasmuch as he has the use of the payment from the time it is paid, should allow credit for the interest he thus gains. The New Hampshire Courts have followed the same general principle with the following modifications: In case the maker of the note makes a payment when no interest of any kind is due, they have ruled that he is simply anticipating a payment due at a future date and that his payment shall draw no interest. This exception they have narrowed by allowing interest in all cases if the payment is large enough to pay the interest due at the end of the year and reduce the principal besides.

SPECIAL PRINCIPLE OF VERMONT RULE.

All payments draw interest from the time they are made to the end of the year.

SPECIAL PRINCIPLE OF NEW HAMPSHIRE RULE.

If no interest is due at the beginning of the year and the payment is not greater than the interest due at the end of the year, the payment draws no interest.

(FACE.)

$100 00. *Chester, Vt., Sept. 6, 1900.*

For value received, I promise to pay J. L. Smith, or order, on demand, one hundred dollars, with interest annually.

S. T. BROWN.

(BACK.)

June 10,1901,
Five dollars.
Dec. 5, 1902,
Twenty-five cents.
Aug. 8, 1903,
Sixty dollars.

SOLUTION.

DATE.	PRIN.	INT. ON PRIN.	INT. ON INT.
Sept. 6, 1900.	$100.00		
Sept. 6, 1901.	100.00	6.00 —5.07	
After Payt.	100.00	.93 12.00	.11 .36
Sept. 6, 1903.	100.00	12.93	.47 —.26
After Payt.	100.00	12.93 18.00	.21 2.33 1.08
Sept. 6, 1906.	100.00 (25.73)	30.93 (56.66)	3.62 —60.28
After Payt.	74.27 1.42		
Jan. 1, 1907.	**$75.69**		

We wish to find the amount due on the preceding note Jan. 1, 1907. As the first step we construct a diagram in which to arrange the principal results obtained in the course of the solution.

How many vertical columns is the diagram made to contain?

What is to be placed in the first column?

In the second? In the fourth?

In the third?

What is first written in the diagram?

What is the date of the first payment?

To what date, then, is interest first computed?

Where do we write this date?

What is the interest on $100 from Sept. 6, 1900 to Sept. 6, 1901?

Where do we write this interest?

What amounts, then, are due before payment Sept. 6, 1901?

From what date to what date do we find the amount of $5, the first payment?

What is its amount?

To what use is it applied?

What sign written before it shows that it is the amount of a payment although written in the "Int. on Prin." column?

How much interest on the principal is due before payment Sept. 6, 1901?

How much after payment?

What are the total amounts due after payment Sept. 6, 1901?

Was any interest due at the beginning of the year in which the $5 was paid?

Is the $5 greater than the interest that became due at the end of the year?

If, then, the note had been given in New Hampshire, would the first payment have drawn interest?

* *

What is the date of the second payment?

What, then, is the second date to which interest is computed?

Where do we write this date?

What kind of interest do we compute on the $.93 due Sept. 6, 1901?

On the $100?

From what date do we compute these interests?

What is the time from Sept. 6, 1901 to Sept. 6, 1903?

What is the simple interest on the $.93 for 2 years?

What is the direct interest on the $100 for 2 years?

The indirect interest?

What is the total interest on interest due Sept. 6, 1903?

The total interest on principal?

What amounts, then, are due before payment Sept. 6, 1903?

From what date to what date do we find the amount of $.25, the second payment?

What is its amount?

To what use is it applied?

What sign is written before it to show it is the amount of a payment?

How much interest on interest is due before payment Sept. 6, 1903? how much after payment?

What are the total amounts due after payment Sept. 6, 1903?

Should we have computed interest on the $.25 if the note had been given in New Hampshire? Why?

* *

What is the date of the third payment?

What, then, is the third date to which interest is computed?

Where do we write this date?

What kind of interest do we compute on the $100 due Sept. 6, 1903? on the $12.93? on tqe $.21?

From what date do we compute these interests?

What is the time from Sept. 6, 1903 to Sept. 6, 1906?

What is the simple interest on the $12.93 for 3 years?

What is the direct interest on $100 for 3 years?

The indirect interest?

What is the total interest on interest due Sept. 6, 1906?

The total interest on principal?

What, then, are the several amounts due before payment Sept. 6, 1906?

From what date to what date do we find the amount of $60, the third payment? What is its amount?

What sign is written before it to show it is the amonnt of a payment?

To what use is it applied?

How much of the amount of the payment remains after paying interest on interest?

What marks are placed about this remainder to show that it is a part of the amount of a payment?

To what use is this remainder applied?

How much of the amount of the payment remains after paying both the interest on interest and the interest on principal?

What marks are placed about this remainder?

To what use is this remainder applied?

What, then, is the amount due after payment Sept. 6, 1906?

What is the date of settlement of the note?

From what date to what date do we compute interest on the $74.27?

What is the amount of $74.27 from Sept. 6, 1906 to Jan. 1, 1907?

What, then, is the amount due on the note Jan. 1, 1907?

$4850. *Claremont, N. H., June 15, 1894.*

For value received, I promise to pay R. T. Eaton, or order, forty-eight hundred and fifty dollars, on demand, with interest annually.

Frank D. Rich.

Indorsements: Apr. 27, 1895, $250; May 20, 1896, $250; Oct. 21, 1898, $1000; Nov. 1, 1899, $100; Feb. 16, 1900, $200; Dec. 3, 1900, $200.

What was due on the preceding note Jan. 1, 1902?

SOLUTION.

DATE.	PRIN.	INT. ON PRIN.	INT. ON INT.
June 15, 1894.	$4850.00		
June 15, 1895.	4850.00	291.00	
		—250.00	
After Payment.	4850.00	41.00	
		291.00	2.46
June 15, 1896.	4850.00	332.00	2.46
		(248.58)	—251.04
After Payment.	4850.00	83.42	
		873.00	15.02
			52.38
June 15, 1899.	4850.00	956.42	67.40
	(15.18)	(971.60)	—1039.00
After Payment.	4834.82		
June 15, 1900.	4834.82	290.09	
	(17.61)	—307.70	
After Payment.	4817.21		
June 15, 1901.	4817.21	289.03	
		—200.00	
After Payment.	4817.21	89.03	
	246.39	157.36	2.91
	2.91	246.39	2.91
Jan. 1, 1902.	**$5066.51**		

EXPLANATION.

We first draw lines so as to form four vertical columns; the first column being for the dates to which the several interests are computed, the second for the principal at the several dates, the third for the interest on principal, and the fourth for the interest on interest.

The first payment was made April 27, 1895. The next June 15 is June 15, 1895. We must, therefore, find the amounts due June 15, 1895.

From June 15, 1894 to June 15, 1895 is 1 year. The interest on $4850 for 1 year is $291. We place this interest in the second column. Therefore, the amounts due on June 15, 1895, are $4850, principal, and $291, interest on principal.

No interest is due at the beginning of the year, and the payment is not greater than the interest due at the end of the year. Therefore, no interest is allowed on $250, the first payment. Applying the $250 to the interest on principal, we find that $41 of interest on principal is still due. Therefore, the amounts due after payment June 15, 1895, are $4850, principal, and $41, interest on principal.

The next payment is made May 20, 1896. The next June 15 is June 15, 1896. We must, therefore, find the amounts due June 15, 1896.

From June 15, 1895 to June 15, 1896 is 1 year. The interest on $41, the interest on principal, for 1 year is $2.46. This we place in the fourth column. The interest on the principal, $4850, for 1 year is $291. This we place in the third column. Adding the amounts in the second column, we find that there are due before payment June 15, 1896, $4850, principal; $332, interest on principal; and $2.46 interest on interest.

There was interest due at the beginning of this year, therefore the payment, though not greater than the interest due at the end of the year, will draw interest. The amount of $250 from May 20, 1896 to June 15, 1896 is $251.04. Applying this amount to the interest on interest, we find that it completely pays it and that a balance of $248.58 remains. Applying this balance to the interest on principal, we find that $83.42 of interest on principal remain due. The amounts due after payment June 15, 1896, are, therefore, $4850, principal, and $83.42, interest on principal.

The next payment is made Oct. 21, 1898. The next June 15 is June 15, 1899. We must, therefore, find the amounts due June 15, 1899. From June 15, 1896 to June 15, 1899 is 3 years. The simple interest for 3 years on $83.42, the interest on principal, is $15.02. This interest we place in the fourth column. The direct interest on the principal, $4850, for 3 years is $873, the indirect interest $52.38. The direct interest we place in the third column, the indirect in the fourth. Adding the amounts in each column, we find that the amounts due before payment June 15, 1899 are $4850, prin-

cipal; $956.42, interest on principal; and $67.40 interest on interest.

Interest was due at the beginning of this year, therefore, the payment, though not greater than the interest due at the end of the year, would draw interest. The amount of $1000 from Oct. 21, 1898 to June 15, 1899 is $1039. Applying this amount to the interest on interest, we find that it pays it, and that a balance of $971.60 remains. This balance we apply to the interest on principal, and find that it pays it, and that a balance of $15.18 remains. This balance we apply to the principal. We thus find that there is due after payment June 15, 1899, $4834.82, principal; no interest on principal; and no interest on interest.

The next payment was made Nov. 1, 1899. The next June 15 is June 15, 1900. We must, therefore, find the amount due June 15, 1900.

From June 15, 1899 to June 15, 1900 is 1 year. The interest on $4834.82, the principal, for 1 year is $290.09. This interest we place in the third column. The amounts due before payment June 15, 1900 are, therefore, $4834.82, principal; and $290.09, interest on principal.

Two payments are made in this year, the later on Feb. 16, 1900. There was no interest due at the beginning of the year, but the sum of the payments is greater than the interest due at the end of the year. The payments, therefore, draw interest. The amount of $100, the fourth payment, from Nov. 1, 1899 to June 15, 1900 is $103.73, and the amount of $200, the fifth payment, from Feb. 16, 1900 to June 15, 1900 is $203.97. Applying $307.70, the sum of these amounts, to the interest on principal, we find that it pays it, and that a balance of $17.61 remains. Applying this balance to the principal, we find that there is due after payment, June 15, 1900, $4817.21, principal; no interest on principal; and no interest on interest.

The next payment was made Dec. 3, 1900. The next June 15, is June 15, 1901. We must, therefore, find the amount due June 15, 1901.

From June 15, 1900 to June 15, 1901 is 1 year. The interest on $4817.21, the principal, for 1 year is $289.03. This we place in the third column. The amounts due before payment June 15, 1901 are, therefore, $4817.21, principal; and $289.03, interest on principal.

No interest is due at the beginning of the year, and the payment is not greater than the interest due at the end of the year. $289.03,

the payment, therefore draws no interest. Applying it without interest to the interest on principal, we find that $89.03 of interest on principal remains due. There is, therefore, due after payment June 15, 1901, $4817.21, principal; $89.03, interest on principal; and no interest on interest.

Complete the explanation,

NOTE. It will be seen that the Vermont and the New Hampshire Rules recognize on the one hand the injustice of allowing the creditor no compensation for failing to receive interest money when due, and on the other hand the fact that the cumulative power of compound interest is so great that under it it would be absolutely impossible in many cases for a debtor to pay a debt that had been running through an extended period.

Each of several payments made within the same year may be less than the interest due at the end of the year, while their sum is greater than that interest. The question whether such payments should draw interest has not been decided by the New Hampshire courts, but in accordance with the opinion of a prominent attorney who has given the subject special attention it is assumed in the preceding solution that they do

Ex. 143.

Find according to the Vermont or the New Hanpshire Rule as your teacher may direct the amont due on each of the following notes:

1. Face of note, $1000; date, Aug. 1, 1900. Indorsements: May 1, 1901, $50; Sept. 1, 1901, $100. Date of settlement, Aug. 1, 1902.

2. Face of note $500; date, Feb. 10, 1902. Indorsements: Sept. 20, 1905, $75.60; May 20, 1907, $200. Date of settlement, Jan. 1, 1912.

3. Face of note, $2200; date, Apr. 4, 1896. Indorsements: May 1, 1897, $90; Nov. 13, 1898, $80; Jan. 1, 1899, $70; Mar. 25, 1904, $70. Date of settlement, Nov. 2, 1904.

4. Face of note, $1000; date Jan. 1, 1890. Indorsements; Sept. 28, 1890, $144; July 17, 1891, $360; Aug. 9, 1891, $160; Sept. 25, 1892, $170; Dec. 11, 1893, $200; July 4, 1895, $75. Date of settlement, June 1, 1897.

5. Face of note, $9000; date Aug. 8, 1893. Indorsements: Oct. 19, 1896, $2500; May 7, 1900, $490; June 20, 1900, $400; Aug. 8, 1900, $50. Date of settlement, Nov. 9, 1904.

Applications of Percentage.

133. Principles and Terms of Percentage.

The principles of Percentage, or of the process of computing by hundredths, differ in no respect from the principles governing the use of other decimals, and have already been developed and applied in this book. As, however, they form the basis of most commercial transactions a more systematic development of them will be given at this point.

1. What is 8% of 32?

2. 45 is 15% of what number?

3. 25 is what per cent of 75?

The three preceding problems represent what are commonly called the **Three Cases** of Percentage. The solutions of the problems, as we have already learned, are as follows:

1. 8% signifies .08. The required number, therefore, is .08 of 32, or 2.56.

2. For every 15 units in 45 there are 100 units in the required number. The required number, therefore, is $\frac{100}{15}$, or 100 times $\frac{1}{15}$, of 45, or 300.

3. 25 is $\frac{25}{75}$, or $\frac{1}{3}$, of 75. $\frac{1}{3}$ expressed as hundredths is $.33\frac{1}{3}$, or $33\frac{1}{3}$%. Therefore, 25 is $33\frac{1}{3}$% of 75.

Percentage is commonly defined as "The process of computing in hundredths." Such a definition is not strictly correct as the addition or the subtraction of hundredths would not commonly be considered exercises in percentage. Instead the following definition may be given:

Percentage is the process employed in solving those problems in which a ratio is a given or a required element, and in which the ratio is expressed or is to be expressed in hundredths.

The following special terms are employed in connection with the problems of percentage:

The number of which a per cent has been taken or is to be taken is called the **Base.**

The per cent that has been taken or that is to be taken is called the **Rate.**

The product that has been obtained or that is to be obtained is called the **Percentage.**

Which element is the base in the first of the three problems? in the second? in the third?

Which element is the rate in the first problem? in the second? in the third?

Which element is the percentage in the first problem? in the second? in the third?

What elements are given and which is required in the first problem? in the second? in the third?

The percentage, evidently, may be thought of as a product, and the base and rate as its factors. Give, then, a rule for finding a percentage when a base and a rate are given; for finding a base when a percentage and a rate are given; for finding a rate when a percentage and a base are given.

Express as a per cent

$\frac{1}{2}$.	$\frac{7}{20}$.	$\frac{2}{3}$.	$\frac{4}{7}$.	$\frac{7}{11}$.	$\frac{2}{7}$.	$\frac{11}{13}$.	$\frac{13}{14}$.
$\frac{1}{4}$.	$\frac{4}{25}$.	$\frac{3}{8}$.	$\frac{3}{5}$.	$\frac{8}{13}$.	$\frac{4}{5}$.	$\frac{14}{17}$.	$\frac{11}{17}$.
$\frac{3}{4}$.	$\frac{9}{40}$.	$\frac{4}{9}$.	$\frac{4}{7}$.	$\frac{7}{15}$.	$\frac{5}{7}$.	$\frac{15}{19}$.	$\frac{17}{18}$.

Express as a per cent

.7.	.137.	.008.	2.7.	.3456.	.0637.
.4.	.216.	.109.	1.04.	.00289.	.582.
.3.	.034.	.904.	5.236.	.0034.	.0364.

Express as a common fraction in its simplest form

$12\frac{1}{2}\%$.	20%.	$33\frac{1}{3}\%$.	40%.	3.5%.
$83\frac{1}{3}\%$.	25%.	$37\frac{1}{2}\%$.	48%.	.46%.
$16\frac{2}{3}\%$.	50%.	$62\frac{1}{2}\%$.	4%.	.024%.

NOTE. The definitions of the terms employed in percentage and the rules for its three cases should be thoroughly mastered, as they are commonly considered to be essential features of the subject. In solving problems, however, work along the lines of the three explanations on the preceding page.

134. Profit and Loss.

An article costing $1.20 is sold at a profit of 33⅓ per cent. What is the gain? What is the selling price?

Give a rule for finding the selling price when the cost and the per cent of gain or loss are given.

* *

An article is bought for $1.20 and sold for $1.80. We wish to find the per cent of gain.

What is the gain?
What is the ratio of $.60 to $1.20?
What is this ratio expressed in hundredths?
What, then, is the per cent of gain?
How would we have found the per cent of loss had the article been sold at a loss of $.60?

Give a rule for finding the per cent of gain or loss when the cost and the selling price are given.

* *

An article is sold at $1.80. The per cent gained is 20. We wish to find the cost of the article?

What is the ratio of the selling price to the cost?
What, then, is the ratio of the cost to the selling price?
How, then, can we find the cost of the article?
What was the cost?
How would we have found the cost if the selling price had been the same and if there had been a loss of 20 per cent?

Give a rule for finding the cost of an article when the selling price and the per cent of loss or gain are given.

* *

A quantity of merchandise is sold at a gain of $120. The per cent of gain is 20. We wish to find the cost.
What is the ratio of the gain to the cost? What, then, is the ratio of the cost to the gain?
How, then, can we find the cost?
What was the cost?

Give a rule for finding the cost when the gain or loss and the per cent of gain or loss are given.

* *

An article is listed by the wholesaler at $3.60. The discounts upon it are 16⅔, 33⅓, 25, and 10 per cent. We wish to find the net price of the article.

What is the price after the first discount? after the second discount? after the third discount? after the fourth discount?

Give a rule for finding the cost of an article when the list price and the several successive discounts are given.

Ex. 1. The list price of a certain quantity of slate-pencils is $12.60. The trade discounts are 10, 25, 30, $33\frac{1}{3}$, and 40 per cent. What is the net price?

Ex. 2. Find the per cent gained on merchandise bought for $75.60 and sold for $96.77.

Ex. 3. The selling price of an article is $48.76. The rate of gain is 35 per cent. What was the cost?

(1)

```
$11.34
  2 83
  8 51
  5 96
  1 99
  3 97
  2 38
```

(2)

```
21.1700   28
15 120
 6 0500
 6 0480
```

(3)

```
4876 |135
405   36.12
 826
 810
  160
  135
   25
```

EXPLANATIONS.

Ex. 1. The price after the first discount is .9 of $12.60, or $11.34; after the second discount, $11.34 less $\frac{1}{4}$ of $11,34, or $8.51; after the third discount, .7 of $8.51, or $5,96; after the fourth discount, $5.96 less $\frac{1}{3}$ of $5.96 or $3.97; and after the fifth discount, .6 of $3.97, or $2.38.

Ex. 2. The gain on the goods is $96.77 less $75.60, or $21.17. The gain, therefore, is $\frac{2117}{7560}$ of the cost. Expressing this gain as a per cent, we find the required per cent to be a very small fraction greater than 28.

Ex. 3. The ratio of the selling price to the cost is $\frac{135}{100}$; therefore, the ratio of the cost to the selling price is $\frac{100}{135}$. The required cost, therefore, is $\frac{100}{135}$ of $48.76, or $36.12.

NOTE 1. Test the result in Ex. 2 by increasing the $75.60 by 28 per cent of itself and comparing the result with the $96.75. Prove in a similar manner all solutions of problems in percentage when you have the least doubt of the correctness of your work.

NOTE 2. The essential point to be comprehended and remembered in connection with Profit and Loss is that a per cent of gain or loss is always based upon the cost, or the money invested.

Ex. 144.

1. A carriage dealer sells 8 carriages at $976, and by so doing makes a profit of 35 per cent. What did he pay for the carriages?

2. A dealer in agricultural implements sells 8 sulky plows and makes $75 profit upon them. His rate of profit is 30 per cent. What was the average price that he paid for the plows?

3. A bookseller sells two sets of books at $25 each. On one set he gains 35 per cent, and on the other set he loses 30 per

cent. Does he gain or lose by both transactions, and what is his gain or loss.

4. $\frac{1}{4}$ of an acre of land is bought at $175 an acre. After being laid out in building lots, $\frac{1}{8}$ of it is given away, $\frac{1}{4}$ is sold at 2 cents a square foot, $\frac{1}{3}$ at 4 cents and the remainder at 8 cents. What is the per cent of profit on the investment?

5. A merchant on Jan. 1, 1901 took an inventory of his goods and found their value to be $17384.73. During the year he purchased goods to the amount of $115834.26, and his sales were $139817.78. His expenses for rent, clerks' salaries, etc., were $9834.27. If the average capital in the business through the year was $65000, and if the value of his stock of goods at the end of the year was $20216.24, what was his per cent of gain?

6. A bill of goods amounting to $175.38 is subject to the following discounts. 10 per cent if paid within 30 days, 20 per cent if paid within 15 days, and 2 per cent additional if cash is sent with order. The goods are sold for $224.30. Find the gain and the per cent of gain if the goods are paid for in 25 days? If they are paid for in 10 days? if cash is sent with the order?

7. A man loans a certain sum of money for a year at 12 per cent and requires the interest to be paid in advance. What per cent of gain is he receiving on the money he invests?

8. A merchant marks a stock of shoes at $2.25. He sells them at a discount of 20 per cent from this price and by so doing makes a profit of 20 per cent. What was the cost of the shoes?

9. A merchant pays $7.50 for a suit of clothes. At what price must he mark them that he may discount 20 per cent from the marked price and still make a profit of 35 per cent?

[What does he wish to receive for the suit? What is to be the ratio of this price to the marked price? What, then, must be the ratio of the marked price to the selling price?]

10. A man's salary at the middle of the year is decreased 10 per cent and at the end of the year is increased 10 per cent. What per cent is the salary that he is receiving at the end of the year of the salary that he was receiving at the beginning of the year?

135. Character of Corporations.

Corporation. A succession or collection of persons having in the estimation of the law existence and rights and duties distinct from those of the individual persons who form it from time to time.—*Law Dictionary.*

It is evident that most branches of business could be carried on to better advantage if life for an unlimited period could be guaranteed to the owners. This advantage is in large degree attained by individuals uniting and by due process of law forming a body called a corporation.

A large and constantly increasing proportion of manufacturing and other industrial enterprises are carried on through corporations. That wealth can be created through such agencies with comparatively small waste of energy is self-evident. The question as regards the consumer of wealth seems to be whether he will receive a considerable share of the saving in the cost of production, or whether the price to him will fail to be decreased through the opportunity being offered to those controlling an article of fixing their price upon it with reference not to its cost but to their own profit.

NOTE 1. A certificate of incorporation is called a **Charter.** The capital invested is made up of a certain number of units called **Shares.** A share is generally $100 but may be any other convenient amount as $50, $25, etc. To collectively designate the shares held in a corporation the term **Stock** is employed. A person holding stock is called a **Stockholder,** and the certificate showing the number of shares held by a stockholder a **Stock Certificate.** Profits, when divided among the stockholders, are referred to as **Dividends** and losses assessed upon them as **Assessments.**

NOTE 2. Corporations are divided into two classes, **Public** and **Private.**

Public Corporations are organizations established by the public for the benefit of the public, such as Nations, States, Counties, Cities, Towns, School Districts, Fire Precincts, Wards, and Incorporated Villages. Such organizations are called Governments. The purpose of a Government in its most limited sense is to repress crime and maintain order within its limits; and in its broader sense to provide for the education, health, and general welfare of its people.

A private corporation is a corporation established for some private

purpose or some public purpose not connected with government. Private corporations include

(1) Corporations established for gain, such as railroads, express companies, and manufacturing companies.

(2) Corporations established for charitable, scientific, educational, and religious purposes, such as hospitals, museums, colleges and churches.

NOTE 2. Railroads, telegraph companies, etc., are in one sense public corporations; that is, they are given by governments the special privilege of right of way, and, in return, are subject to government control. Thus, a state may fix the maximum rates that shall be charged by any railroad, express company, etc., within its limits.

NOTE 3. The character of public corporations, and of private corporations established for gain, is indicated by the following extracts taken from the Vermont statutes:

"The inhabitants of every town or school district are hereby declared to be a body politic and corporate, and by their corporate names may sue and be sued, prosecute and defend any proper action or suit, by an agent or attorney, chosen for that purpose.—Chap. 83, Sect. 3.

"The term 'private corporation,' as used in this chapter, shall be considered to mean any corporation created for the purpose of making a turnpike road, railroad, or canal, for carrying on any brauch of manufacture, for mining, for improving navigation of any stream or other waters, for building wharves or store-houses, for building or using steamboats or other vessels, for the purpose of banking or insurance, and all other corporations which, from their object, suppose a division of profits among their stockholders.—Chap. 86, Sect. 2.

NOTE 4. When stock can be sold at its face value it is said to be **at Par;** when above its face value, above par; and when below its face value, below par.

136. Notes and Bonds.

Promissory Note. An absolute promise in writing, signed but not sealed, to pay a specified sum at a time therein limited, or on demand, or at sight, to a person therein named or designated, or to his order, or to the bearer.—*Law Dictionary.*

Bond. A contract under seal to pay a sum of money.—*Law Dictionary*

Near the end of this book will be found a copy of a blank stock certificate issued by the Hillside Creamery of Cornish, N. H., and a copy of a bond issued by the Town of Newport, N. H. Study these papers until you thoroughly understand them and are able to answer without hesitation the following questions.

By what corporation was the stock certificate issued?

What is the par value of each share of the stock?

How are the stock certificates of this corporation transferable?

NOTE. There is an important peculiarity in the operation of this and most of the other creamery corporations in Vermont and New Hampshire. In place of the profits or losses being divided among the stockholders, each stockholder receives annually 6 per cent interest on the money he has invested, and the remaining income, less the current expenses and the expenses for repairs, new machinery, etc., goes to the patrons who furnish the creamery with milk. The stock certificates of such creameries, therefore, are practically 6 per cent bonds, and, as might be expected, the price at which they are ordinarily sold is near their par value.

By what corporation was the bond issued?

What kind of a corporation is the Town of Newport?

Under the laws of what larger corporation was it established?

To whom is the bond payable?

What is the par value?

In what kind of money is it payable?

When must the Town of Newport pay the bond?

Where is it payable?

What interest does it bear? Where and when is this interest payable? In what kind of money is it payable? By what must an application for an interest payment be accompanied?

When may the Town of Newport pay any bond?

How must it advertise its intention to exercise this privilege?

When must the advertisement be issued?

What is the penalty of failing to present a bond that is called for payment?

How was the issue of the Newport Water Bonds authorized? When was it authorized?

By whom is this bond signed? by whom is it countersigned? What one act has made it a bond instead of an ordinary note?

Write a stock certificate of an imaginary manufacturing corporation to be established in your town Write a municipal gas bond.

137. Selling Price of Stocks.

The capital of a certain local telephone company is $10,000. Its net earnings the first year are $1200.

What is the income on each $100 invested?

What would be the income on $100 at 6 per cent?

What, then, is the ratio of the income on money invested in the stock to the income on the same amount of money loaned at 6 per cent?

What price, then, assuming that the current rate of interest in the locality is 6 per cent, could one afford to pay for a $100 share of the stock?

What price could one afford to pay if the income from each $100 share was

12 dollars?	15 dollars?	2 dollars?
48 dollars?	8 dollars?	0 dollars?

Give a general principle for determining the market value of stocks.

* *

Suppose that the indications were that the business of the company would constantly increase.

What effect would these indications have upon the selling price of the stock?

Suppose that a report was circulated that a rival line was to be established. What effect would this report have upon the selling price of the stock?

Give some of the conditions under which the selling price of stock is greater or less than its earning capacity.

NOTE 1. The term "Stocks" in financial reports includes both bonds and shares in corporations, and persons buying and selling stocks for others are called stock-brokers. Brokers' commissions are commonly computed on the par value of stocks and are usually ⅛ per cent.

The term "Bond" is applied not only to a promissory note given under seal but also to a sealed document guaranteeing the fulfilment of an obligation or providing for restitution in case of misappropriation of funds. Thus, a contractor may be required to give a bond for the completion of a building before a certain date or a public official may be required to secure "bondsmen" before entering upon the performance of his duties.

NOTE 2. The value of bonds depend

(1) Upon the rate of interest paid upon them. Thus, on Aug. 12, 1902, U. S. 3's are quoted at 105¼, and U. S. 4's at 132.

(2) Upon the time the bonds have to run. Thus, in the same

report in which U. S. 4's are quoted at 132, U. S. old 4's, which are payable in a comparatively short period are quoted at only 108¼.

(3) Upon the financial responsibility of the persons or corporations by whom they are payable. Thus, in the report in which U. S. 4's are quoted at 132, C. B. & Q. 4's are quoted at 96 and Mex. Cen. 4's at 82⅛.

138. Problems in Stocks.

A man buys 10 shares of stock at 7⅛ per cent premium and pays ⅛ per cent brokerage. We wish to find what capital he invests?

What does he pay for one share?

How, then, may we find what he pays for 10 shares?

Give, then, a rule for finding the cost of any quantity of stock.

A man has a certain amount to invest in stocks at 12 per cent discount. We wish to find how many shares he can buy, brokerage being ⅛ per cent.

What must he pay for one share?

How, then, may we find how many shares he can buy?

Give, then, a rule for finding the number of shares of stock that can be bought with a given number of dollars.

A man buys stock at 98⅞ and sells it at 93⅛. He loses by the investment $110. We wish to find how many dollars he invests, brokerage each way being ⅛ per cent.

What does each share cost him?

What does he receive for each share?

What, then, does he lose on each share?

How, then, may we find the number of shares in his investment?

How may we find his investment?

Give, then, a rule for finding the amount invested when the gain or loss and the cost and the selling price of each share are given.

A certain stock pays a dividend of 7 per cent. We wish to find the par value of the stock held by a man who receives a dividend of $126.

What dividend does he receive on each share?

How, then, may we find the number of his shares?

How, then, may we find the par value of his stock?

Give. then, a rule for finding the par value of stock when a dividend or an assessment and the rate of dividend or assessment are given.

We wish to find what rate of income will be realized on 5 per cent bonds bought at 125.

What is the income from each share?

What is the cost of each share?

How, then, may we find the rate of income?

Give, then, a rule for finding the rate of income on bonds bearing a given interest and bought at a given price.

A town issues 5 per cent bonds to pay a certain debt. It wishes to pay only 4 per cent interest on the money borrowed. We wish to find at what price it must dispose of the bonds.

What is the ratio of 4 to 5?

What, then, must be the ratio of the face of the bonds, or the debt on which interest is to be paid, to the selling price of the bonds, or the money that the town receives?

What, then, must be the ratio of the selling price of the bonds to the face of the bonds?

Give, then, a rule for finding the price at which bonds bearing a given rate of interest must be sold by the maker that a certain given rate of interest may be paid on the money received from them.

NOTE. The remaining cases in stocks and bonds are so similar to the preceding that a development of their principles would seem to be unnecessary. Whatever the problem all that one need do is to consider a share the unit and proceed accordingly.

Ex. 1. At what price must 4 per cent bonds be bought to pay an annual income of 5 per cent?

Ex. 2. A man buys stock at 115. The first two years it pays 7 per cent and the third year 6 per cent. Immediately after receiving the third diivdend he sells the stock at 107. What is the average per cent of annual income?

Ex. 3. A man buys stock at 101¾ and sells it at 99⅜, brokerage each way being ⅛ per cent. He loses $189. What was his investment?

Ex. 4. What is the interest on a $1000 U. S. 3 per cent bond from Jan. 1, 1902 to July 1, 1902?

SOLUTIONS.

(1)

80

(2)

```
400 | 115
345 |3.48
 550
 460
  900
```

(3)

$189 \times \frac{8}{21} = 72$

101⅞
815
7335

(4)

```
180
  1
181
54.30 | 365
365   |12.14
 780
 730
  500
  365
  135
```

EXPLANATIONS.

Ex. 1. Each $100 bond will produce an income of $4. This income is to be 5 per cent, or $\frac{5}{100}$, of the cost of the bond. The cost of the bond, therefore, is $\frac{100}{5}$ of $4, or $80.

Ex. 2. The dividends on each share are 7 plus 7 plus 6, or 20, dollars. The final loss on each share is 115 less 107, or 8, dollars,

thus reducing the net income to 20 less 8, or 12, dollars. The average annual income, therefore, is \$4, and the average rate of income \$4 divided by \$115, or $.03\frac{1}{23}$, or a little less than 3.48

Ex. 3. The cost of each share, including the brokerage, is \$101⅛. The selling price, less the brokerage, is \$99¼. The loss on each share, therefore, is \$101⅛ less \$99¼, or \$2⅝, and the number of shares is \$189 divided by \$2⅝, or 72. The amount invested, therefore, was 72, or 9 times 8, times \$101⅛, or \$7335.

Ex. 4. From Jan. 1, 1902 to July 1, 1902 is 6 months. Three of these months, January, March, and May have 31 days, and one, February, has only 28 days. The exact number of days, therefore, between the two dates is (6 times 30) plus (3 less 2), or 180 plus 1, or 181, days. The required interest, therefore, is $\frac{181}{365}$ of .05 of \$1000, or \$12.14.

Ex. 145.

1. A man owns stock whose par value is \$3500. It is assessed at 18 per cent. What assessment does he pay?

2. A man owns stock on which a dividend of 6 per cent is declared. The amount of his dividend is \$306. Find the par value of his stock.

3. A man invests \$3028 in stock quoted at 94⅛, brokerage being ⅛ per cent. Find the par value of his stock.

4. A man buys 4 per cent stock at 103. Find the rate of income on his investment.

5. Which is the better investment, and by what per cent, 4 per cent bonds bought at 112, or 3½ per cent bonds bought at 101?

6. Which would be the better investment, U. S. 4's payable in 12 years at 115 or U. S. 4's payable in 5 years at 104?

[Take into consideration the fact that at maturity only the par value of the bonds will be paid.]

8. Standard Oil stock is quoted on Aug. 12, 1902 at 668. Assuming that the average earning capacity of money through the United States is 6 per cent, about what per cent in dividends is paid annually by the Standard Oil Company?

9. Adams Express Co. stock on Aug. 12, 1902, is quoted at 205, American at 243, United States at 124, and Wells-Fargo at 218. Find the approxinate per cent in dividends paid by each express company, assuming that the average earning capacity of money is 6 per cent.

No.......... BOND. $100.

TOWN OF NEWPORT,

SULLIVAN COUNTY, NEW HAMPSHIRE,

Per 3 Cent.

WATER BOND.

FOR VALUE RECEIVED, the Town of Newport, a municipal corporation legally established under the laws of the State of New Hampshire, acknowledges itself indebted to the bearer hereof in the sum of

ONE HUNDRED DOLLARS

payable in legal tender of the United States of America, and promises to pay said sum to the bearer at the office of its Treasurer on the first day of August, nineteen hundred and twenty-one, at three per cent. per annum, payable in like legal tender at the same place on the first day of February, 1902, and on the first day of August and February thereafter, in each year, upon the presentation of the coupons hereto annexed as they severally mature, with the right reserved to said Town of Newport at its option to pay the principal of this bond at any time on or after the first day of August, 1909. Notice of exercise of said option shall be by publication in two newspapers published in the County of Sullivan, State of New Hampshire, three weeks successively, the last publication to be fourteen days at least before such time of payment, and when notice has been given as aforesaid, interest on the bonds so called shall cease from and after the day named for payment.

The issue of bonds of which this bond is a part was authorized by vote of the Town of Newport at a town meeting legally held for that purpose March 12, 1901.

IN TESTIMONY WHEREOF the said Town of Newport has authorized this instrument to be signed by its Selectmen, countersigned by its Treasurer, and its corporate seal to be affixed thereto this ________ day of 1901.

139. Banks and Bank Paper.

Banking. The maintenance of a place of business for the receiving and the loaning of money constitutes the nature and function of banking.

The preceding definition, taken from a standard work on banking, draws a clean-cut line between banking and ordinary loaning of money. The division of banks into classes is hardly as simple a matter, but the most serviceable classification is, perhaps, the following:

1. **Banks of Issue.** These banks, like all other banks, receive deposits from those who have more money on hand than they require for their immediate needs, and make loans to those who need more ready money than they have in their possession and who can give security for the payment of the money that they borrow.

In addition, they issue notes signed by themselves payable on demand. These notes, as explained on page 224, are not legal tender. They circulate, however, among the people and draw interest the same as the other forms of money.

The only banks that issue notes in the United States at the present time are the **National Banks.** The conditions essential to the formation of such banks are as follows:

1. At least five persons must form the organization.

2. The capital of a bank must be at least $25,000, and banks can be established with this capital only in places of 3000 population or less.

3. A certain part of the capital stock of each bank must be in United States Bonds. These bonds must be deposited with the United States Treasurer at Washington, and when so deposited the Government will print and forward notes up to their par value. The minimum deposit of bonds is $12,500, and the minimum for banks with a capital not less than $50,000, ¼ of the capital stock of the bank.

4. Banks pay a tax of ½ of 1% on circulation secured by 2% bonds, and 1% on circulation secured by any other bonds.

NOTE 1. The bonds belonging to the banks held by the United States form absolute security against loss by holders of bank notes.

Interest, of course, is paid on these bonds to the banks by the Government.

NOTE. 2. Previous to the creation of the present National Banks

in 1863, notes were issued by state banks. A national tax of 10 % was then laid upon these banks, which, of course, made it impossible for them to continue their business as banks of issue.

NOTE 3. Deposits in National Banks are principally made by business men, who thus secure a place for the safe keeping of their money and an opportunity to use it, through the drawing of checks, with even greater convenience than though it were in their personal possession. Such deposits do not ordinarily draw interest.

2. Savings Banks. Loans are rarely made by National Banks for a longer period than 6 months, and the security required for a loan is either the indorsement of the note by a second person of sound financial standing, or collateral security in the form of stocks, bonds, etc., that have a recognized value in the general money market. Banks of another class called **Savings Banks,** make loans for extended periods and largely upon real estate as security. Scuh banks also differ from National Banks in that certain restrictions are placed upon the withdrawal of deposits. In compensation, however, interest, usually from 2 to 4 per cent, is always paid depositors.

The deposits in savings banks are principally made by people of moderate means. In consideration of this fact, in most States they are indirectly given certain special privileges and are subject to certain special restrictions. Thus, in New Hampshire, the deposits in savings banks pay only ¾ of 1% tax, a rate much less than the average tax upon individual property; and, on the other hand, special statutes restrict the loaning of the funds of the banks within certain definite limits.

3. Private Banks. The following extracts from Chapter 164 of the New Hampshire Statutes show the nature of private banks and of the regulations that control them. The heading of the chapter is, "Bank Cashiers and Private Banks."

As previously explained, the existence of state banks as banks of issue was terminated by the legislation establishing the National Banks.

Section 1. The cashier of every state bank shall, on the first Monday of March, June, September, and December, in each year, make a statement of its condition that day, specifying in separate columns the capital stock actually paid in; debts due the bank secured by pledge of its stock; value of real estate belonging to the bank;

amount of debts due from directors, either as principals or sureties, specifying whether on interest or otherwise; amount of specie in the vault; amount of bills of other banks on hand; amount of deposits in the bank; amount of deposits in other banks for the redemption of its bills; and the amount of bills of the bank then in circulation.

Section 5. Every association or partnership formed for the purpose of loaning money or dealing in money, receiving deposits, buying and selling exchange, or transacting such other business as is usually transacted by banks, shall be a bank for the general purposes of the title and for taxation.

Read carefully the copies of blank notes given near the end of the book:

The one who gives a note is called the **Maker**, and the one to whom it is paid the **Payee**. Any person who agrees to become responsible for the payment of a note is called an **Indorser**.

A note payable at a specified time is called a **Time** note, and a note payable on demand of the payee a **Demand** note. A note signed by two or more persons each of whom becomes responsible for the entire amount of the note is called a **Joint and Several** note.

NOTE. A note or a check made payable to the order of the original payee may be transferred an indefinite number of times. In this case each payee becomes an indorser to each of the following payees. Thus, suppose that B, who holds a note signed by A, transfers it to C, and that C transfers it to D. In this case A is the maker of the note, and D the final payee; C is an indorser to D, and B is an indorser to both C and D. C and D can, however, relieve themselves of their liability by writing after their indorsement the words "without recourse" or words of similar import.

If a note secured by indorsers is not paid when it becomes due notice of a specified legal form must be sent within a specified time to each depositor, or he will be freed from any obligaton to pay it. Such a notice is called a **Protest.**

A creditor frequently collects a debt that is over-due by sending a notice to the debtor directing him to pay the amount of the debt to the creditor himself through a local bank, or to some third party. Such an order is called a **Sight Draft.** If the draft is an order to pay the debt within a specified number of days, it is called a **Time**

Draft, and becomes, when accepted in writing by the debtor, practically a time note.

The term "draft" is also applied to an order by one bank, signed by its cashier, upon a second bank which has in its possession funds belonging to the first bank.

* *

What kind of a note is the first? who is the payee? Where is the note payable? Upon what security would this note be accepted?

What kind of a note is the second? the third? the fourth? Explain the sqecial characteristics of each note; of the check.

Who is the drawer of the draft? the drawee? the payee?

Explain each of the indorsements.

Write a time note; a joint and several demand note: a check; a time draft.

140. Problems in Bank Discount.

When money is borrowed from National Banks, interest is ordinarily paid in advance. Thus, a person who gives a note at a bank for $100 to be paid in 60 days receives from the bank $100 less the interest on $100 for 60 days, or $99.

When money is thus loaned the amount named in the note is called the Face of the note; the amount received by the borrower is called the Proceeds; and the interest paid in advance is called the Bank Discount.

If A holds a time note against B, A can discount the note at a bank at any date between the date of the note and the date at which it is payable. In this case, the bank computes the exact number of days between the date of discount and the date of maturity and reckons each day as $\frac{1}{360}$ of a year

* *

The only problem in bank discount that presents any special difficulty is the finding of the face of a note when the proceeds and the other essential elements are given. That the solution of this problem employs no new principle will be seen from the following inductive exercises.

A man wishes to borrow money from a bank for 60 days at 6 per cent with which to pay a debt of $100. We wish to find the amount for which the note must be given.

What is the interest on $1 for 60 days at 6%?

What, then, will be the proceeds of $1 for the given time and rate?

What will be the ratio of the proceeds to the face of the note?

What, then, will be the ratio of the face of the note to the proceeds?

Give a rule for finding the face of a note when the proceeds, the rate, and the term of discount are given.

Ex. 1. What is the face of a note for 135 days whose proceeds at 6% are $576.34?

Ex. 2. A note for $275.34 dated March 19, 1902, and bearing 5% interest is payable March, 1, 1903. If it is discounted at a bank Oct. 20, 1902 what will be its proceeds?

SOLUTIONS.

(1)

```
                100·00
                 02 25
5763400·        |97 75·
48875           |589 61
 87590
 78200
  93900
  87975
   59250
   58650
     600
```

(2)

```
 275·34
  13 767
     688     131
  13 08        1
 288 42      132
 576 84
6·345 24
 282 07
```

EXPLANATIONS.

Ex. 1. The interest on $1 at 6 per cent for 135 days is .0225. The proceeds of the note, therefore, are 100 less 2.25, or 97.75, hundredths, or $\frac{9775}{10000}$, of the face of the note, and the face of the note is $\frac{10000}{9775}$ of the proceeds. $\frac{10000}{9775}$ of $576.34 is $589.61. The required face of the note, therefore, is $589.61.

Ex. 2. From March 19, 1902 to March 1, 1903, is 1 year less 18 days.

The interest on $275.34 for 1 year less 18 days is .05 of $275.34 less ¼ of .01 of $275.34, or $13.767 less $.688, or $13.08. The sum that the bank will receive when the note becomes due is, therefore, $275.34 plus $13.08, or $288.42.

From October 20, 1902 to March 1, 1903 is 4 mo. 11 da.

Three of these months, October, December, and January have 31 days, and one, February, has only 28 days. The number of days, therefore, between the date of discount and the date of maturity is (4 times 30) plus 11 plus (3 less 2), or 120 plus 11 plus 1, or 132. The bank discount on $288.42 for this time is .022 of $288.42, or .001 of 11 times 2 times $288.42, or $6.35; and the proceeds are $288.42 less $6.35, or $282.07.

Ex. 146.

1. Find the proceeds of a note for $269.34 payable in 60 days and discounted at a bank at 6 per cent.

Find the proceeds if, as is the law in some States, 3 days of grace are added to the time named in the note.

2. Find the face of a note for 90 days whose proceeds when discounted at a bank at 5 per cent are $450.

3. July 6, 1902, A gave B a note for $237.26 payable in seven months without interest. Sept. 1, B discounted the note at a bank at 5 per cent. What were his receipts from the note?

4. Apr. 18, 1902, C gives D a note for $800 payable in 6 months with interest at 6%. What will C receive for the note if he discounts it at a bank on Aug. 8? What will he receive if he discounts it the day it is given him?

5. A note of $150 is payable without interest in 6 months. Find the difference between the time discount and the bank discount on the note, the rate in each case being 6%.

Deposits in Savings Banks.

The payments of interest made to depositors in savings banks are commonly called **Dividends.** The regulations governing the payment of dividends vary slightly in different banks. The following extracts, however, from the by-laws of the Sugar River Savings Bank, of Newport N. H., together with the accompanying problem and solution, will illustrate the general principles upon which the regulations of the different banks are based.

SECTION 1. On the second Tuesday of May and November in each year there shall be declared a dividend at the rate of 4 per cent. per annum upon all sums which shall have been deposited for the term of six months next preceding the first day of the month in which the dividend is declared, and at the same rate upon all sums deposited during the six months for each and every whole month the same have been deposited reckoning from the first day in each month, provided, however, that if the net earnings will not allow a dividend at that rate then at such rate as the trustees may order. Dividends will be entered to the depositor's credit and if not withdrawn put upon interest.

SEC. 2. In case deposits are made immediately before the first day of April, or are withdrawn soon after said date, so as to cause the bank to bear an undue proportion of the taxes, the officers of the bank reserve the right to deduct such amount, on account of such taxes, from the deposit as may be just and equitable.

SEC. 3. No interest will be paid on any sums withdrawn for the term which may have elapsed since the last dividend, nor shall any interest be paid on sums withdrawn before any dividend shall have been declared on them.

The interest paid by the Sugar River Savings Bank at the present time is 3 per cent. A depositor has in the bank on May 1, 1902 $756.34. During the ensuing year he deposits in the bank and draws from the bank the sums shown in the first column of the following diagram. We wish to find the amount to his credit on May 1, 1903.

SOLUTION.

DEPOSITS AND AMOUNTS WITHDRAWN.	DATES OF DIVIDENDS.	NET DEPOSITS.	INTEREST.
May 28, 1902, $75;	Nov. 1, 1902.	$756.34	$11.34
July 10, 1902, $24.75;		4.75	.03
Sept. 1, 1902, $30;		761.09	
Sept. 15, 1902, —$50;		11.37	
	New Principal.	772.46	
Nov. 2, 1902, $80;	April 1, 1903.	752.46	11.28
Dec. 20, 1902, —$100;		75.00	.38
Feb. 10, 1903, $75;		827.46	
		11.66	
	New Principal.	839.12	

EXPLANATION.

$50 is withdrawn from the bank Sept. 15. This counterbalances the deposit of $30 made Sept. 1, and leaves but $4.75 of the deposit made July 10. The amount to the credit of the depositor on Oct. 1, 1902 is, therefore, $756.39 plus $4.75, or $761.09.

$756 of the $756.34 draw interest from May 1 to Aug. 1, or for 6 months. This interest is .015, or 15 times .001, of $756, or $11.34.

$4 of the $4.75 draw interest from Aug. 1 to Nov. 1, or for 3 months. This interest is ¾ of 1% of $4, or ¾ of $.04, or $.03. The total amount, therefore, to the credit of the depositor on Oct. 1, 1902, is $11.34 plus $.03, or $11.37, plus $761.09, or $772.46.

Complete the explanation.

A depositor on Nov. 1, 1902 had to his credit in the Sugar River Savings Bank $843.29. His deposits during the following year were Dec. 1, 1902, $175.85; Feb. 20, 1903, $80; Apr. 1, 1903, $120; May 25, 1903, $45.64; Aug. 1, 1903, $25; Sept. 15, 1903, $48. He drew out of the bank during the year the following amounts: Dec. 15, 1902, $25; Jan. 1, 1902, $150; Apr. 1, 1903, $50; July 1, 1903, $150; and Aug. 20, 1903, $40. What amount was to his credit in the bank Nov. 1, 1903?

Time Note.

$ .... *East Jaffrey, N. H.,* *190*

For value received, ... *promise to pay to the order of the Monadnock National Bank of East Jaffrey, the sum of* .... *Dollars* ....

$\frac{}{100}$

after date .... *payable at said Bank.*

....

....

Joint and Several Demand Note.

$ .... *Newport, N. H.,* *190*

For value received, we, each as principal, jointly and severally, promise to pay the First National Bank of Newport, or bearer, at said bank, ... *Dollars, on demand, with interest after* .... *days.*

And we also agree that said bank may enforce, delay or extend payment of this note at pleasure without affecting our liability thereof.

....

....

....

Savings Bank Note.

$ .... *East Jaffrey, N. H.,* *190*

For value received, I promise to pay the Monadnock Savings Bank, or order, ... *Dollars, on demand, with interest semi-annually at* ...... *per cent. per annum, the interest payable on the first days of June and December in each year.*

....

....

No. ....

COLLATERAL SECURITY NOTE.

$ ....   NEWPORT, N. H., ...190

....For value received..............................promise

to pay..or order

....and interest at the rate of........................per centum per

annum ....for such time as said principal sum, or any part shall remain unpaid

..................................having deposited with this obligation as COLLATERAL SECURITY..

....

....

....

and should the market value of same decline,..................promise to furnish satisfactory additional collateral on demand; and..................hereby give authority to sell the same, and also any collaterals substituted for or added to the above, without notice, either at public or private sale, or otherwise, at the option of the holder hereof, on the non-performance of either of the above promises, and also before the maturity of this note, if the market value of the collaterals should decline, the holder hereof giving........................credit for any balance of the net proceeds of such sale remaining, after paying all sums due from........................on account of this obligation, and it is further agreed that said collaterals may be applied at any time towards any obligation of............................to said Bank, with like authority to sell the same upon breach of any such obligation, without notice, and upon the terms and conditions aforesaid; and it is hereby agreed that the holder or holders of this note, or any person in his or their behalf, may purchase at any such sale.

....

141. Exchange.

A lives in Boston, and B and C live in New York. A owes B \$100, and C owes A \$100. How can A pay B without sending money from Boston to New York?

A bank in Boston keeps a definite sum of money on deposit in a bank in New York. How may a person in Boston pay a debt that he is owing a person in New York?

The payment of a debt by one person to another through the agency of a third is called **Exchange.** The first and the second parties to the transaction may, evidently, be the same person, that is, a person starting on a trip to a distant city or country may purchase from a bank in the town in which he lives an order payable to himself upon a bank in the city or country he is to visit.

Exchange is principally carried on (1) through the **Mail,** (2) through **Express Companies,** and (3) through **Banks.**

1. Exchange through the United States Mail. Money can be sent through the mail (1) by **Registered Letters,** and (2) by **Money Orders.** The Government will not be responsible for more than \$25 sent in a registered letter, and will not issue a money order for a greater amount than \$100.

2. Exchange by Express Companies. Money may be sent through express companies (1) by money orders, and (2) by **Travelers Cheques,** which "Are practically Certified Cheques payable to one's own order, in Gold or its equivalent, by upwards of 10,000 Correspondents throughout the world." The rates for money orders are the same as for postal money orders, and for travelers checks approximately the same as the cost through banks.

3. Exchange Through Banks. The price of domestic exchange through banks is not uniform. The price of Foreign Exchange on Aug. 29, 1902 is shown by the following table sent on that date to the First National Bank of Newport, N. H.

Ex. 147.

1. Find the face of the check that \$125 will purchase in exchange on each of the countries named in the table on the following page.

2. Find the number of pounds sterling in a 60 days bill of exchange that can be purchased for \$125; the nnmber of francs; the number of marks.

Exchange.

Fees Charged for Issuing International Orders, in Effect July 1, 1901.

DOMESTIC RATES.		INTERMEDIARY RATES.		INTERNATIONAL RATES.	
WHEN PAYABLE IN CANADA, CUBA, AND THE PHILIPPINES:		WHEN PAYABLE IN MEXICO:		WHEN PAYABLE IN ANY OTHER FOREIGN COUNTRY:	
The Domestic form must be used for these Orders.		*Use the International Form.*		*Use the International Form.*	
For Order not exceeding $2.50,	$0.03	For Order not exceeding $10,	$0.05	For Order not exceeding $10,	$0.10
From 2.50 to 5,	.05	From 10 to 20,	.10	From 10 to 20,	.20
" 5 to 10,	.08	" 20 to 30,	.15	" 20 to 30,	.30
" 10 to 20,	.10	" 30 to 40,	.20	" 30 to 40,	.40
" 20 to 30,	.12	" 40 to 50,	.25	" 40 to 50,	.50
" 30 to 40,	.15	" 50 to 60,	.30	" 50 to 60,	.60
" 40 to 50,	.18	" 60 to 70,	.35	" 60 to 70,	.70
" 50 to 60,	.20	" 70 to 80,	.40	" 70 to 80,	.80
" 60 to 75,	.25	" 80 to 90,	.45	" 80 to 90,	.90
" 75 to 100,	.30	" 90 to 100,	.50	" 90 to 100,	1.00

Rates of Foreign Exchange, August 29, 1902.

FOR CHECKS:	Under $50.	$50 to $500.	$500 to $10,000
Pounds Sterling,	4.88½	4.87¾	4.87⅛
Turkish Pounds,	4.50	4.49	4.48
Francs,	19.43	19.40	19.38
Lei on Roumania,	19.75	19.65	19 55
Kronen on Austria-Hungary,	20.44	20.40	20.37
Finmark on Finland,	19.45	19.42	19 40
Lire, paper, on Italy,	19.33	19.30	19.25
Marks on Germany,	23.90	23 83	23.81
Kroner on Norway, Denmark, and Sweden,	26.92	26.87	26 83
Gulden on Holland,	40 33	40.30	40 25
Mexican Dollars on Mexico,	43.25	44.00	43.50
Dollars on Manila,	44.25	44.00	43.50
Dollars on Hong Kong,	43 75	43.50	43.00
Yen on Yokohama,	51.75	51.50	51.25
Rubles on Russia and Poland,	51.80	51.75	51.60
Rupees on India,	33.65	33.40	33.30

Checks in United States Dollars on America and Hawaii ½%, on Philippine Islands ¾%.

FOR PURCHASE.

	Under $500.	$500 to $10,000.	60 Day Bills.
Pounds Sterling,	4.84	4 86	4.83½
Francs,	5.21⅞	5.18¾	5.21⅞
Marks on Germany,	.94	.94[illegible]	.94¼

142. Present Worth and True Discount.

We wish to find the present value of a debt of $150 due in 6 months, the current rate of interest being 6 per cent.

How can we find the sum of money that loaned at 6 per cent will amount to $150 in 6 months?

How, then, shall we find the present value of the debt?

How can we find the deduction that the creditor can afford to make for the debt's being paid at the beginning of the 6 months?

Give a rule for finding the present value of a debt due at some future time.

Give a rule for finding the discount that should be made for the payment of a debt before it becomes due.

The sum of money which paid at the present time will justly pay a debt due at some future time is called the **Present Worth** of the debt, and the deduction made on account of such payment is called the **True Discount.** Problems in Present Worth and True Discount, as has been shown in the preceding inductive exercises, are simply problems in finding a principal when an amount is given and in comparing this principal with the given amount.

Ex. 148.

1. What is the present worth on May 1, 1902 of a debt of $324.86 payable Jan. 10, 1903, the current rate of interest being 6 per cent?

2. Find the present worth, at 6 per cent annual interest, on Sept. 16, 1900 of $125 due Feb. 8, 1904.

3. A man offers to sell a horse for $150 cash or for $160 payable without interest in 6 months. If the current rate of interest is 6 per cent which is the better bargain for a purchaser, and how much?

4. The price of a set of books sold on instalments is $12, the terms of payment being $1 on the delivery of the books and $1 at the end of each month. If the purchaser can invest his money at 6 per cent interest to what cash price is this price equivalent?

fered a discount of 5 per cent for cash payment. If he cannot

5. A man owing a debt of $300 payable in 8 months is of-
invest his money at better than 6 per cent what will he gain by accepting the offer?

6. Find the true discount, at 6 per cent, on a debt of $148.60 due Dec. 8, 1901 if paid July 7, 1901.

143. Insurance.

Insurance. An agreement between two persons (the insurer and the insured) that in consideration of a comparatively small payment (called the "premium") by the insured the insurer will, on a certain event happening during a given time, pay to the insured either an agreed sum or the amount of the loss caused to the insured by the event.—*Law Dictionary.*

The nature of insurance is made clear by the preceding definition. Its classes, on the basis of the character of the event whose occurrence is the condition of the paying of the indemnity, are numerous. Among the principle classes are insurance against death, or **Life Insurance**; insurance against accident, or **Accident Insurance**; insurance against fire, or **Fire Insurance**, and insurance against loss at sea, or **Marine Insurance.**

The processes employed in preparing, for instance, life insurance tables are (1) determining by statistics the average number of persons in a thousand who die at the age of each number of years between certain limits; (2) fixing the premium for each age so that the premiums at that age of all the insured shall pay the insurance of every one; and (3) increasing these premiums by a certain per cent to provide for salaries, commissions to agents, general expenses, and profits for the company.

While most other states allow insurance companies to pay, when to their advantage, only the actual value of property distroyed by fire, New Hampshire obliges them to pay the full amount for which they have insured it and on which they have received premiums from the insured.

Ex. 149.

1. A house worth $2400 is insured at $\frac{3}{4}$ of its value at $\frac{1}{8}$ of 1% a year. What will be the cost of the insurance at the end of 5 years (1) allowing no interest on the premiums, (2) allowing simple interest on the premiums, (3) allowing annual interest, (4) allowing compound interest.

2. A house is insured for $\frac{2}{3}$ of its value, and the rate of insurance is $\frac{1}{4}$%. If the annual premiums are $6 what is the value of the house?

3. Learn the amount and the rate of the insurance on your school-house. Find the annual premium paid for insurance.

144. State and Municipal Taxes.

Money raised for the support of a public corporation is called a **Tax.** Taxes are commonly divided into **Direct Taxes** and **Indirect Taxes.** Direct taxes include **Poll** Taxes, or taxes assessed upon persons without reference to their property; **Real Estate** Taxes, or taxes assessed upon land, buildings, etc.; and **Personal Property** Taxes, or taxes upon all forms of property outside of real estate.

Among the taxes assessed by a State and paid directly to the State Treasurer are **Charter Fees,** fees paid for the grant or the renewal of charters to corporations; **Corporation** Taxes, or taxes upon certain corporations within the limits of the State; and **License** Fees, or fees for carrying on certain lines of business.

Taxes assesssed by towns, usually at their annual town meeting, commonly include a **State** Tax, a **County** Tax, a **Town** Tax, a town **Highway** Tax, and a town **School** Tax. Towns in some States vote the amount of taxes they will raise, and in others a certain rate of taxation.

145. United States Taxes. The taxes laid for the support of the National Government are **Duties,** or taxes laid upon imported goods, and **Internal Revenue,** or taxes laid upon the manufacture of tobacco and liquors, upon licenses to sell liquors, etc. Most countries also lay taxes upon the incomes of their citizens, but this mode of taxation has been declared unconstitutional by a recent decision of the United States Supreme Court.

Ex. 150.

1. A town votes to raise a tax of $7500. There are in the town 205 male citizens between 21 and 70 years of age, each of whom pays a poll tax upon $100. The value of the real estate of the town is $423850, and of the taxable property $104283. A rate of what integral number of thousandths will raise the required tax?

2. What amount should the assessors use in their estimate in place of the $7500 if a discount of 5 per cent is allowed all tax-payers who pay all their taxes before a certain date?

3. B, a citizen of the town, is 52 years old. His real estate is valued at $3400, and his taxable personal property at $1257. What will be the amount of his taxes?

146. Equation of Payments.

A merchant on June 1, owes a wholesaler $150 due in 15 days, $200 due in 28 days, and $300 due in 48 days. We wish to find when without loss or gain of interest he can pay the three debts by a single payment of $650.

Suppose that he pays the $650 on June 1. For how many days will he lose interest on the $150?

The interest on $150 for 15 days is equivalent to the interest on $1 for how many times 15 days?

How, then, shall we find the number of days' interest on $1 that he will lose by paying the $150 on June 1, instead of when it becomes due?

How shall we find the number of days' interest on $1 that he will lose by paying the $200 on June 1? by paying the $300 on June 1?

How shall we find the total number of days' interest on $1 lost by paying the three amounts on June 1, instead of on the dates at which they are due?

The merchant, evidently, must keep the $650 long enough after June 1, to gain the interest he would lose by paying it on that date.

What is the ratio of the number of days required for $650 to gain a certain interest to the number of days required for $1 to gain the same interest?

Explain, then, the final step in the solution of the problem.

Give in order the several steps in finding the date at which payment may be made, without loss of interest to debtor or creditor, of several debts due at different future dates.

Ex. 1. A is owing B $75.86 due in 24 days, $116.25 due in 33 days and $204.39 due in 69 days. When can he pay the three debts by a single payment without loss or gain of interest?

Ex. 2. A debt of $425 is due May 9, 1902. $265 are paid Jan. 10, 1902. When should the balance be paid?

SOLUTIONS.

(1)

1046·25 183951
9197 55
1024380 | 396·50
79300
231380
26
24
60

(2)

31800
3153·5
197

Nov. 22, 1902.

EXPLANATIONS.

Ex. 1. We assume that the entire debt should be paid in 24 days. The first debt is due in 24 days, and by paying it at that time the debtor would lose no use of his money.

The second debt is not due for 33 days, and by paying it in 24 days the debtor wonld lose the use of $116.25 for 33 less 24, or 9, days; or the use of $1 for 9 times 116.25, or 1046.25, days.

The third debt is not due for 69 days, and in paying it in 24 days the debtor would lose the use of $204.39 for 69 less 24, or 45, days; or the use of $1 for 45, or 5 times 9, times 204.39, or 9197.55, days. The total use of money lost by him, therefore, would be the use of $1 for 1046.25 plus 9197.55, or 10243.80, days. Instead, therefore, of paying the $496.50 in 24 days he should keep it long enough after the expiration of 24 days for its use to equal the use of $1 for 10213.80 days, This time, evidently, is 10243.80 divided by 396.50, or 26, days. The payment should be made, therefore, in 24 plus 36, or 60, days.

Ex. 2. In paying the $265 on Jan. 10, 1902 instead of on May 9, the date at which the debt is due, the debtor loses the use of $265 for 119 days, or the use of $1 for 119, or 120 less 1, times 265, or 31800, days. He must, therefore, keep $160, the balance of the debt, long enough after May 9 for its use to equal the use of $1 for 31800 days, or for 31800 divided by 160, or 197, days.

197 days are 4 months and 17 days. 6 months from May 9 is Nov. 9. But 4 of the 6 months between these dates have 31 days each. The required date, therefore, is 6 months and 17 less 4, or 13, days after May 9, 1902, or Nov. 22, 1902.

Ex. 151.

1. A owes B $65 due in 28 days, $90 due in 43 days, and $150 due in 62 days. Find the equated time for paying the debts by a single payment.

2. A owes B $316.34 due Mar. 16, $486.24 due July 8, $132 due Aug. 6, and $83.25 due Nov. 10. What is the equated time for payment?

3. A merchant owes a wholesaler $134 due in 36 days, $208 due in 45 days, and $75.24 due in 52 days. What sum paid in 45 days will pay the debt?

[Find first the equated time for the payment of the debts.]

4. A merchant owes $450.74 due May 16, 1901. Jan. 3, he pays $375. When should he pay the balance?

5. Find the equated time for paying $75 due in 45 days, $23.75 due in 50 days, $83.24 due in 19 days, $103.25 due in 38 days, and $148 due in 72 days.

6. A owes B $75 due in 40 days, and B owes A $100 due in 75 days. What will be the equated time for B to pay A the balance of $25?

147. Commission and Brokerage.

Stocks as we have learned, are usually bought and sold through brokers, who receive as compensation a per cent of the par value of the stock called **Brokerage.** In like manner, commodities of nearly every description are bought and sold for a percentage of the money invested or received called **Commission.** The only principle of any difficulty that appears in commission or brokerage is the one developed in the following inductive exercises.

A commission merchant receives a certain amount of money to invest in goods after deducting his commission of 3 per cent on the goods that he is to purchase. We wish to find his commission and the number of dollars that will remain with which to purchase goods.

What per cent of the money invested is the agent to retain as his commission?

What, then, will be the ratio of the money iuvested plus the commission, or of the total amount of money received, to the money invested?

What will be the ratio of the money invested to the total amount of money received?

How, then, shall we find the amount to be invested?

Give, then, a rule for finding the amount to be invested when the sum remitted for the purchase and the commision is given.

SOLUTION.

```
110000 | 41
 82      2682.93
 280        67 07
 246
  340
  328
   120
    82
    380
    369
     110
```

A commission merchant receives $2750 to invest in goods after deducting his commission of $2\frac{1}{2}$ per cent. What amount can he invest?

EXPLANATION.

The commission is to be $2\frac{1}{2}$ per cent of the money invested. The money remitted, therefore, must be 100 plus $2\frac{1}{2}$, or $102\frac{1}{2}$, per cent, or $\frac{205}{200}$, of the money invested, and the money invested must be $\frac{200}{205}$ of the money remitted. The money invested, therefore, will be $\frac{200}{205}$, or $\frac{40}{41}$, of $2750, or $2682.93, and the agent's commission will be $2\frac{1}{2}$ per cent, or $\frac{1}{40}$, of $2682.93, or $67.07.

NOTE. When more convenient, find the agent's commission by subtracting the money invested from the money remitted.

Ex. 152.

1. A commission merchant receives $400 to invest after deducting his commission of 3 per cent. What sum can he invest, and what will be his commission?

2. An auctioneer sells a quantity of goods at 8 per cent commission and receives as his share of the proceeds $47.52. What was received for the goods?

3. A commission merchant who has sold a quantity of goods retains his commission of 4 per cent and remits to the owner as the net proceeds $184.76. What was received for the goods?

[What is the ratio of the amount remitted to the owner to the amount received for the goods? What, then, is the ratio of the amount received to the amount remitted?]

4. A commission merchant sells goods for $1346.84 and invests the proceeds in sugar at $4\frac{11}{16}$ cents a pound. If his commission for each transaction is $2\frac{1}{2}$ per cent, how many pounds of sugar will he buy?

148. Relation of the Means of a Proportion to the Extremes.

[Read carefully Notes 1 and 2 on page 48. What definition do you find given of a ratio? of a proportion?]

The first and the last terms of a proportion are called the **Means**, and the two middle terms the **Extremes**.

We wish to find the relation between the product of the means and the product of the extremes.

The first three terms of a proportion are 4, 5, and 6. What must be ratio of the fourth term to the third term?

What is the ratio of the first term to the second term?

How will the product of $\frac{5}{4}$ of 6 by $\frac{4}{5}$ of 5 compare with the product of 6 by 5?

How, then, will the product of the extremes in the complete proportion compare with the product of the means?

A ratio with its terms inverted is called an **Inverse Ratio.** What kind of a ratio is $\frac{4}{5}$ as compared with $\frac{5}{4}$?

Suppose the ratio of the fourth term of a given proportion to the third term to be any fraction, however simple or complex. What will be the ratio of the first term to the second term?

What is the product of any ratio and its inverse ratio?

How, then, must the product of the means of any proportion compare with the product of the extremes?

How, then, may we find either extreme when the means and the other extreme are given. How may we find either mean when the extremes and the other mean are given?

NOTE. The preceding laws, though, commonly given as the fundamental principles of proportion, are unnecessary and of little practical value. Suppose, for instance, that we wish to find the missing

term in the proportion 8 : 14 : : ( ) : 10. As a proportion is an equality of ratios the third term must be $\frac{8}{14}$, or $\frac{4}{7}$, of the fourth, or the same part of the fourth that the first is of the second.

The first term of a ratio is called an **Antecedent,** and the second term a **Consequent.**

What are the consequents in the preceding proportion? the antecedents?

149. To Solve Problems by Proportion.

Five men can dig a ditch 125 rods long in 8 days. How many rods can 40 men dig in the same time?

Five men can dig a certain ditch in 8 days. How many days will it take 20 men to dig the same ditch?

NOTE. The solution of the preceding problems is exceedingly simple. Thus, in the first problem, the work done by 40 men will evidently be $\frac{40}{5}$, or 8, times the work done by 5 men, and the number of rods dug will be 8 times 5, or 40. In the second problem, as 20 men will in the same time do $\frac{20}{5}$, or 4, times as much work as 4 men, the time required for them to do the given amount of work will evidently be only $\frac{5}{20}$, or $\frac{1}{4}$, as great, or 2 days. That is, the problems are solved by an application of the simple principle of ratio, which forms the basis of the solution of all problems solved by a multiplication or a division. It is, however, commonly considered essential for a student of mathematics to have the capacity to solve such problems by forming from them a proportion with one missing term, and then finding the value of that term.

To solve problems in proportion pursue the following course of reasoning and operation.

1. Think of the missing element of the problem and of the corresponding given element as results, or Effects, and of the remaining elements as conditions, or Causes. Thus, in the problem whose solution is given on the following page, the required principal and the given principal are thought of as effects, and the remaining elements as causes.

2. Think whether the proportion is direct or inverse, that is, whether a change in a cause produces a like or an opposite change in the effect. To determine this point, commence your reasoning with the equivalent of "The More," and see whether the effect must be preceded by "the more" or "the less." Thus, the more interest to be gained the greater the

principal required to gain it --a direct proportion; but the greater the rate the less the principal required to gain a given interest —an inverse proportion.

If $300 at 6 per cent will gain $12 interest in 8 months, what principal will be required at 8 per cent to gain $25 in 10 months?

SOLUTION.

(1)

prin.	r.	int.	t.
300	6	12	8
( )	8	25	10

$$\left.\begin{array}{r} \cancel{4} : \cancel{3} \\ \cancel{12} : 25 \\ \cancel{5} : \cancel{4} \end{array}\right\} :: \overset{15}{\cancel{300}} : (375)$$

(2)

$$\$300 \times \tfrac{3}{4} \times \tfrac{25}{12} \times \tfrac{4}{5} = \$375$$

EXPLANATION.

1. The term having no corresponding term given is $300, the given principal. The $300, therefore, we regard as the first effect, and the required principal as the second effect, The first causes, evidently, are 6 per cent, $12 interest, and 8 months, the conditions under which $300 principal will be required; and the second causes, 8 per cent, $25 interest, and 10 months, the conditions under which the missing principal will be required.

The greater the rate the less the principal that will be required to gain a given interest. Our first proportion, therefore, is inverse, and we state it: 8 per cent, the second cause, is to 6 per cent, the first cause, as $500 principal, the first effect, is to the required second effect; or, simplifying the first ratio, 4 : 3 :: 300 : the missing term.

The greater the interest to be gained the greater the principal required to gain it. Our second proportion, therefore, is direct, and we state it: $12 interest, the first cause, is to $25 interest, the second cause, as $300 principal, the first effect, is to the required second effect.

Explain the statement of the third proportion.

The ratio of the second term of the proportion to the first is the ratio of $(3 \times 25 \times 4)$ to $(4 \times 12 \times 5)$, and this ratio must also be the ratio of the fourth term to the third. Multiplying the $300 by this ratio, we find the fourth term of the proportion, and the required principal, to be $375.

2. A principal will gain $\frac{8}{6}$ as much at 8 % as at 6 %, therefore the principal required to gain a given interest will be only $\frac{6}{8}$, or $\frac{3}{4}$, as great, and we must multiply the given principal by $\frac{3}{4}$.

The principal required to gain $25 will be $\frac{25}{12}$ of the principal required to gain $12. We must, therefore, multiply $\frac{3}{4}$ of $300 by $\frac{25}{12}$.

Complete the explanation.

NOTE 1. As the first step in solving a problem in proportion, it will be found profitable to write down its essential elements according to the method illustrated in the preceding solution. Moreover, it will be advisable, for obvious reasons, to write the given effect with its causes in the first line although the causes of the missing effect appear first in the problem.

NOTE 2. A ratio that is the product of several simple ratios is called a **Compound Ratio,** and a proportion containing a compound ratio is called a **Compound Proportion.** Thus, the ratio of the required principal to the given principal in the problem just explained is a compound ratio. and the proportion by which the missing term is obtained is a compound proportion.

If necessary, multiply both terms of a ratio by such a number as will change them to integers. Thus, the ratio $3\frac{1}{4} : 2\frac{1}{6}$ should be mentally changed to $39 : 26$, or $3 : 2$.

Observe that the cancellations in the solution were, (1) the cancellation of a 3 and a 4 from factors of the dividend and of a 12 from a factor of the divisor. and (2) the cancellation of a 4 and a 5 from factors of the divisor and of a 20 from a factor of the dividend.

Ex. 153.

1 If 7 men can dig a ditch 75 rods long in 1 day, how long a ditch can be dug in 8 days by 14 men?

2. If 9 men working 9 hours a day can dig a ditch 230 rods long, $2\frac{1}{2}$ feet wide, and 4 feet deep in 4 days, how long a ditch 3 feet wide and $4\frac{1}{2}$ feet deep can be dug in 9 days by 5 men working 10 hours a day?

3. If 40 men can build 150 rods of wall in 40 days, how many men can build 200 rods in 24 days?

4. If 6 men dig a ditch in 22 days of 9 hours each, in how many days of 10 hours each can 9 men perform the same work?

5. If 15 horses consume 40 tons of hay in 10 months, how much hay will 13 horses consume in the same length of time?

6. When 7 bushels of wheat can be bought for \$6.50, how many bushels can be bought for \$25?

7. If 13 barrels of flour are required for a family of 8 persons for 18 months, how many barrels will be required for a family of 14 persons for 12 months?

8. If a bin 12 feet high and 4 feet wide will hold 300 bushels

of corn, what must be the width of another bin that is of the same length and 10 feet high to hold 350 bushels?

8. If a building lot 75 feet wide and 100 feet deep costs $500, how deep must a lot 100 feet wide be to cost $750?

10. If a certain sum of money at 8 per cent will gain $370 interest in 2 yr. 6 mo., in what time will the same sum at 5 per cent gain $290 interest?

11. If a wheel 6 ft. 5 in. in circumference and making 100 revolutions a minute will pass over a certain distance in 2 h. 34 min., what must be the circumference of a wheel making 110 revolutions a minute to pass over the same distance in 1 h. 54 min.?

12. If the value of a pile of wood 6 feet high, 4 feet wide, and worth $4.50 a cord is $75, what must be the height of a pile of the same length 16 inches wide and worth $6.50 a cord to have a value of $50?

150. Problems in Partnership.

A, B, and C, form a partnership. A invests $1500, B $2500, and C $2000. Their net gain for the year is $1600. We wish to find each partner's share of the gain.

What is the total investment?
A's investment is what part of the total investment?
A's gain, then, is what part of the total gain?
How, then, shall we find A's share of the gain?

How shall we find B's share of the gain? C's share? Give a rule for the division of the gain or loss among two or more partner- when the capital of each is invested for the same time.

The principles underlying the process of finding each partner's share of the gain or loss when the capital of each is invested for a different time are illustrated by the following solution and explanation.

A starts in business on Jan. 1, with a capital of $2000. May 1, he receives B as a partner, who invests $1500. July 1, C becomes a member of the firm and invests $2500. The gain at the end of the year is $819.76. We wish to find each partner's share of the gain.

SOLUTION.

24000
12000
15000
51000
$ 48.22 $\frac{2}{17}$
385.77
192.88
241.11

EXPLANATION.

A loses the use of $2000 for 12 months, or of $1 for

2000 times 12, or 24000, months; B loses the use of $1500 for 8 months, or of $1 for 1500 times 8, or 12000, months; C loses the use of $2500 for 6 months, or of $1 for 2500 times 6, or 15000, months; and the three lose the use of $1 for 24000 plus 12000 plus 15000, or 51000, months.

A's loss of the use of his money is $\frac{24}{51}$, or $\frac{8}{17}$, of the total loss; B's loss is $\frac{12}{51}$, or $\frac{4}{17}$, of the total loss; and C's loss is $\frac{15}{51}$, or $\frac{5}{17}$, of the total loss. A's share of the gain, therefore, is $\frac{8}{17}$ of $819.76, or $385.77; B's share is $\frac{4}{17}$, of $819.76, or $192.88; and C's share is $\frac{5}{17}$ of $819.76, or $241.11.

Ex. 154.

1. A and B hire a pasture for $75. A puts in 7 cows for 20 weeks, and B puts in 8 cows for 17 weeks. What must each pay as his share?

1. A, B, C, and D form a partnership. A invests $2100, B $1800, C $2500, and D $2800. At the end of the year their net gain is $1451.27. What is each partner's share?

2. Jan. 1, A engages in business with a capital of $8700. Apr. 15, B becomes a partner in the business and invests $10000. June 1, C enters the firm with a capital of $6000. Their net gain for the year is $2708.24. What is each partner's share of the gain?

3. Jan. 1, four men form a partnership. A invests $3000 but on June 1 draws out $800. B invests $2500 and on June 15 invests an additional $500. C invests $4000 and on Aug. 10 draws out $1000. D invests $3500. July 15, he draws out $500 and Oct. 10 he invests an additional $200. The net gain for the year is $1874.27. Find each partner's share of the gain.

4. A, B, C, and D form a partnership. Jan. 1, A invests $4000, B $4500, C $3600, and D $5000. A draws out $200 on May 15, $140 on Aug. 10, and $175 on Oct. 1. B draws out $180 on Apr. 15, $200 on July 15, and $125 on Oct. 8. C draws out $250 on June 1 and $175 on Sept. 15. D draws out $50 on Feb. 12, $100 on May 1, $150 on June 1, and $200 on Oct. 1. After paying all liabilities the resources at the end of the year are cash $6218.27, notes due the firm $4234.96, and merchandise to the wholesale value of $7854.47. It is agreed that the gain due each partner shall be credited to him and made a part of his capital for the following year. What will be each man's share of the gain, and with what capital for the following year will he be credited?

Square and Cube Root.

[Read carefully Notes 4, 5, 6, and 7 on pp. 47—48.]

What is the square of 5? the cube of 4? the fourth power of 3?

Define a power; a square; a cube.

$9^2 = ?$ $5^3 = ?$ $2^4 = ?$ Define an exponent.

What is the square root of 49? the cube root of 64? the fifth root of 243?

Define a root; a square root; a cube root.

$\sqrt{64}$ equals what? $\sqrt[3]{64}$ equals what?

Define an index; a radical sign. Explain the signification of the radical sign without an index.

151. Square Root a Special Case in Factoring.

It is evident that one who has a thorough aquaintance with the multiplication and division tables up to 12 times 12 can name instantly the square root of any perfect square not greater than 144. The square root of any larger power can, of course, be found by squaring supposed roots until the correct one is obtained. Thus, the square root of 576 is evidently between 20, whose square is 400, and 30, whose square is 900. Moreover, assuming that 576 is a perfect square, the root must be either 24 or 26 as these are the only numbers between 20 and 30 whose squares contain 6 as the right-hand figure. As 576 is much nearer 400 than 900 we may feel certain that its root is 24 rather than 26. Squaring 24, we obtain 576. The required root, therefore, is 24.

It is evident that this process in the case of large numbers would be tedious and unscientific. But by raising a number to the second power, and indicating each operation instead of performing it, we discover a relation so simple between a square and its root that the pupil cannot fail to comprehend the principle underlying each step in the reverse process of extracting the root from the power.

Summary of Facts. The facts that we discover are as follows:

1. That a square is made up of a certain number of products,

2. The location of these products, and, in consequence, their number.

3. Of what each product is composed.

From these facts it follows that square root is simply a special case in factoring, and that the logical place for the treatment of square root is with the other cases under factoring and directly following multiplication and division.

152. To Square a Binomial.

What is the first step in the multiplication of 23 by 23? the second? the third? the fourth?
What is the first step in the addition of the partial products? the second? the third?

We wish to square the expression 20+3, and to indicate each multiplication instead of performing it. The parts in such an expression connected by the plus sign are called **Terms**, and expressions of two terms are called **Binomials.**
What will be the first step in multiplying (20+3) by (20+3)? the second? the third? the fourth?

What is the simplest expression that will represent the product of 3 by 3? that will represent the product of 20 by 20 ? How shall we represent the product of 20 by 3? How will the product of 3 by 20 compare with the product of 20 by 3? How, then, may we represent the product of 3 by 20?

What will be the first step in adding the partial products? the second?
Once any expression plus once the expression is how many times the expression? What, then, will be the second term of the complete product?
What will be the third step in adding the partial products?

Raise 20+3 to the second power, indicating each multiplication instead of performing it, and explain each step of the operation.

In the same manner raise to the second power

40 + 5	860 + 5	218 + 2	404 + 9
45 + 8	260 + 3	199 + 6	560 + 2
30 + 5	370 + 2	325 + 8	197 + 3

* *

We have a number of four figures that we wish to raise to the second power. We will think of the plus sign as expressed before the units and treat the part of the number preceding the sign as a single term. We will, moreover, indicate the four multiplications instead of performing them.

$$\begin{array}{r} \text{prfs} + \text{u} \\ \underline{\text{prfs} + \text{u}} \\ (\text{prfs} \times \text{u}) + \text{u}^2 \\ \underline{\text{prfs}^2 + (\text{prfs} \times \text{u})\phantom{+\text{u}^2}} \\ \text{prfs}^2 + 2(\text{prfs} \times \text{u}) + \text{u}^2 \end{array}$$

What abbreviation do we use in the multiplication to represent the units? to represent the figures that precede the units?

What expression do we use to represent the product of the units by the units? to represent the product of the preceding figures by the units? to represent the product of the units by the preceding figures? to represent the product of the preceding figures by the preceding figures?

Explain the addition of the partial products?

What expression do we obtain for the square of a number whose last figure is a unit?

We observe that the first term of the expression that we have obtained is "prfs^2." What must be the order of the last figure of the figures preceding the units?

How shall we express these figures as a binomial?

Our next step, evidently, is to square the preceding figures plus the tens, or the expression (prfs + t). Square this expression in the same manner that you squared the expression (prfs + u). What expression do you obtain?

What is the first term of the expression for the square of (prfs + t)?

What is the order of the last figure of the figures preceding the tens?

How shall we express these figures as a binomial?

Square the expression (prfs + h). What expression do you obtain?

Of what order is the first term of this expression?

What, then, in our last product may we write in place of prfs^2?

Using the abbreviation t th for ten-thousands, .t for tenths, .h for hundredths, etc., raise to the second power an expression for a number whose last figure is

Of thousand's order.	Of tenth's order.
Of ten-thousand's order.	Of hundredth's order.
Of million's order.	Of thousandth's order.

* *

$$\begin{array}{l} \text{th}^2 \\ + \text{h}(2\,\text{prfs} + \text{h}) \\ + \text{t}(2\,\text{prfs} + \text{t}) \\ + \text{u}(2\,\text{prfs} + \text{u}) \end{array}$$

We have found that the expression in the margin is the expression for the square of a number of four integral orders. Write the expression for the square of a number of three integral orders; of five integral orders; of two decimal orders; of two integral and two decimal orders; of one integral and three decimal orders; of three integral and two decimal orders; of four integral and four decimal orders.

153. To Find by What a Figure of Any Order Is Multiplied when a Number Is Raised to the Second Power.

By what must 5 be multiplied to produce 85? to produce 90? By what binomial must it be multiplied to produce the binomial (85 + 90)?

An expression consisting of a single term is called a **Monomial.** Separate each of the following binomials into a monomial and a binomial factor. Make the monomial factor as large as possible.

25 + 35.	32 + 36.	125 + 150.
48 + 64.	81 + 108.	144 + 180.
50 + 75.	60 + 45.	200 + 250.

An expression of three terms is called a **Trinomial.** We wish to factor the last two terms of the trinomial $\text{prfs}^2 + 2(\text{prfs} \times \text{u}) + \text{u}^2$.

What common monomial factor have these two terms?

$$\text{prfs}^2 + 2(\text{prfs} \times \text{u}) + \text{u}^2 = \text{prfs}^2 + \text{u}[2\,\text{prfs} + \text{u}]$$

What is the second principle of multiplication?

How, then, do we divide the expression $2(\text{prfs} \times \text{u})$ by u?

What factors remain after the division?

What, then, is the quotient of $2(\text{prfs} \times \text{u}) \div \text{u}$?

We must next divide u^2 by u. u^2 signifies the units taken how many times as a factor?

What is the quotient of $(\text{u} \times \text{u}) \div \text{u}$?

What, then, is the quotient of $\text{u}^2 \div \text{u}$?

What expression, then, do we obtain in place of $2(\text{prfs} \times \text{u}) + \text{u}^2$?

By what, then, are the units multiplied in raising any number to the second power?

Factor in the same manner the last two terms of the square

Of (prfs + t).	Of (prfs + .t).	Of (prfs + h).
Of (prfs + .h).	Of (prfs + h th).	Of (prfs + .m).

As a result of your factoring by what do you find the tens to be multiplied in squaring a number? by what the hundreds? the thousands? the tenths? the hundredths? the millionths?

By what, evidently, is each figure of a number multiplied when the number is raised to the second power?

154. Number and Location of the Products.

In squaring a number by what are the units multiplied?

With what figure of the power, then, must the units' product end?

By what are the tens multiplied?

In multiplying tens by tens how many times is the tens' figure taken as a factor?

How does the order of each of these factors compare with the order of the corresponding factor in the units' product?

How, then, will the order of the tens' product compare with the order of the units' product?

With what figure of the power, then, will the right-hand figure of the tens' product end?

By similar reasoning explain with what figure of the square the product by hundreds will end.

The product by thousands.

The product by tenths.

The product by hundredths.

The product by thousandths.

If, then, a dot is placed over the unit figure of a square and over each second figure at its left and its right, each dot will represent the position of what?

How, then, will the number of dots compare with the number of products in the power?

With the number of figures in the root?

Give, then, a rule for finding the number of figures in the root of a given square.

NOTE. We have learned that a tens' product, for example, ends two orders at the left of the right-hand figure of the square. Observe, however, that it begins at least as many places at the left as there are figures in the root. Observe the effect of this fact upon the process of extracting a root.

Summary of Principles. The process of extracting the square root of a number is made evident by the relations that we have discovered between a square and its root. For convenience of reference, however, we here summarize the facts and principles on which the several steps of the process are based.

1. In squaring a number each figure is multiplied by twice the preceding figures plus itself. Every square, therefore, may be thought of as composed of as many products as there are figures in its root.

2. The right-hand product of a square ends with the right-hand figure of the square, and each of the remaining products ends two figures at the left of the adjacent right-hand product.

3. A zero must be annexed to the preceding figures in each product that they may be of the same order as the common factor of the last two terms.

EXPLANATION.

We are to extract the square root of 1398.76.

We first place a dot over the 8 units and over each second figure at the left and at the right of the 8 units. We thus find that the root of 1398.76 will consist of three figures, and that these figures will be tens, units, and tenths. An expression, therefore, of the relation between 1398.76 and its root will be $t^2 + u(2\,\text{prfs} + u) + .t(2\,\text{prfs} + .t)$

SOLUTION.

$$1\dot{3}9\dot{8}.\dot{7}6 = t^2 + u\,(2\,\text{prfs} + u) + .t\,(2\,\text{prfs} + .t)$$

```
      1398.76 (37.4
     7 9
   60)498
     4 469
  740)2976
      2976
```

We have learned that the square of the left hand figure of the root is found in the left hand period of the power. 13 is not a perfect square, the square cannot be greater than 13, therefore it must be the greatest square less than 13, or 9. If 9 is the square of the tens, the tens themselves will be the square root of 9, or 3. Subtracting this square and annexing to the remainder the next period, we have as the tens' product 498.

We have learned that the greater part of this product is made up of the units multiplied by twice the preceding figures plus the units. The greater part of the second factor is twice the preceding figures. Therefore, if we divide by twice the preceding figures we shall get the units or a number a little larger than the units. Dividing by twice the preceding figures, first reducing them to the order of the figure by which they are to be multiplied by considering a 0 to be annexed, we obtain as the quotient figure 7. But the second factor is not made up of twice the preceding figures alone, but of twice the preceding figures plus the units. Therefore, to complete the second factor we must add the units. Adding, multiplying by the units, subtracting. and bringing down the next period, we have for a new product 2976.

Let the pupil complete the explanation.

NOTE 1. If a number is not a perfect square express it as a mixed number, and extract its root to the required number of decimal places.

NOTE 2. The product of any number by 0 is 0. Therefore, in case a figure of the root is found to be 0 it is unnecessary to complete the divisor and multiply by the root figure. Instead, write the 0 as a figure of the root, annex the next period to obtain a new dividend, and then proceed as though the root figure had been a significant figure instead of a 0.

NOTE 3. In deciding upon a quotient figure, make allowance for the necessary increase in the divisor. This increase will be proportionally greatest in the first divisor, and will become less and less of an element with each following division.

NOTE 4. To extract the root of a fraction whose numerator and denominator are perfect powers, extract the root of the numerator and of the denominator. Thus, $\sqrt{\frac{9}{16}}=\sqrt{9}\div\sqrt{16}=3\div4=\frac{3}{4}$.

To extract the root of a fraction whose denominator is not a perfect square, reduce the fraction to a decimal and extract the root of the decimal. Thus, $\sqrt{\frac{4}{7}}=\sqrt{.5714}=.75+$

If the denominator is a perfect square but the numerator is **not**, either extract the root of the numerator to a sufficient number of decimal places and divide it by the root of the denominator, or reduce the fraction to a decimal and extract the root of the decimal.

Thus, $\sqrt{\frac{237}{625}}$ to 2 decimal places $=15.39\div25=.61+$, or $=\sqrt{.3792}=.61+$

Ex. 155.

1. $\sqrt{1225}=?$
2. $\sqrt{7396}=?$
3. $\sqrt{2304}=?$
4. $\sqrt{5041}=?$
5. $\sqrt{53.29}=?$
6. $\sqrt{70.56}=?$
7. $\sqrt{94.09}=?$
8. $\sqrt{42.25}=?$
9. $\sqrt{50.41}=?$
10. $\sqrt{10.24}=?$
11. $\sqrt{105625}=?$
12. $\sqrt{5959.84}=?$
13. $\sqrt{10.4976}=?$
14. $\sqrt{552.25}=?$
15. $\sqrt{238144}=?$
16. $\sqrt{9.3025}=?$
17. $\sqrt{180625}=?$
18. $\sqrt{71.0649}=?$
19. $\sqrt{294849}=?$
20. $\sqrt{8.6436}=?$

Ex. 156.

Extract to three decimal places the square root of each of the following numbers:

1.	54	13.	3.25	25.	9415
2.	26	14.	4.60	26.	7684
3.	42	15.	5.13	27.	91.27
4	93	16.	7.16	28.	9.035
5.	74	17.	84.2	29.	214.3
6.	65	18.	.936	30.	7.945
7.	52	19.	.034	31.	91.26
8.	15	20.	9.07	32.	434.5
9.	41	21.	9.24	33.	68.67
10.	75	22.	7.08	34.	.8376
11.	58	23.	2.76	35.	.8629
12.	57	24.	9.42	36.	.5246

155. To Cube a Binomial.

We wish to raise a number whose last figure is of units' order to the second power.

What expression have we obtained as the square of $(\text{prfs}+\text{u})$?

$$\begin{array}{r} \text{prfs}^2 + 2(\text{prfs}\times \text{u}) + \text{u}^2 \\ \text{prfs} + \text{u} \\ \hline (\text{prfs}^2 \times \text{u}) + 2(\text{prfs}\times \text{u}^2) + \text{u}^3 \\ \text{prfs}^3 + 2(\text{prfs}^2\times \text{u}) + (\text{prfs}\times \text{u}^2) \qquad \\ \hline \text{prfs}^3 + 3(\text{prfs}^2\times \text{u}) + 3(\text{prfs}\times \text{u}^2) + \text{u}^3 \end{array}$$

To obtain the cube of $(\text{prfs}+\text{u})$ we, evidently, must multiply its square by $(\text{prfs}+\text{u})$. What is the first step in this multiplication? the second? the third? the fourth ? the fifth? the sixth?

Explain the first multiplication; the second; the third; the fourth; the fifth; the sixth.

Explain the addition of the partial products.

In the same manner raise to the third power the expression

prfs + t.	prfs + h th.	prfs + .h.	prfs + h.
prfs + m.	prfs + .th.	prfs + th.	prfs + .t.

156. To Find by What a Figure of Any Order is Multiplied when a Number Is Raised to the Third Power.

We have obtained as an expression for the cube of a number whose last figure is of units' order the expression "the cube of the preceding figures, plus 3 times the square of the preceding figures multiplied by the units, plus 3 times the preceding figures multiplied by the square of the units, plus the cube of the units," or $\text{prfs}^3 + 3(\text{prfs}^2 \times \text{u}) + 3(\text{prfs}\times \text{u}^2) + \text{u}^3$. We wish to factor the last three terms of this expression so as to find by what the units are multiplied in raising a number to the third power.

How do we remove a monomial factor from a binomial?

$$\begin{array}{l} \text{prfs}^3 + 3(\text{prfs}^2\times \text{u}) + 3(\text{prfs}\times \text{u}^2) + \text{u}^3 \\ \text{prfs}^3 + \quad \text{u}[3\,\text{prfs}^2 + 3(\text{prfs}\times \text{u}) + \text{u}^2] \end{array}$$

How, evidently, shall we remove a monomial factor from a trinomial?

Explain the removing of the factor u from the second term of the cube of $(\text{prfs}+\text{u})$; from the third term; from the fourth term?

By what do you find that the units are multiplied in raising a number to the third power?

Factor in the same way the last 3 terms of the expression for the cube

Of prfs + t.	Of prfs + t th.	Of prfs + .h.
Of prfs + h.	Of prfs + m.	Of prfs + .th.
Of prfs + th.	Of prfs + .t.	Of prfs + .t th.

By what do you find that the tens are mnltiplied in raising a number to the third power? by what the hundreds? the thousands? the ten-thousands? the tenths? the hundredths? the thousandths? the millionths?

By what, evidently, is each figure of any number multiplied in raising the number to the third power?

We have found that the expression for the cube of a number of four integral orders is the expression shown in the margin. Write the expression for the cube of a number of 3 integral orders; of 8 decimal orders; of 3 integral and 3 decimal orders.

$$\begin{array}{l} th^3 \\ +h[3\,prfs^2+3(prfs\times h)+h^2] \\ +t[3\,prfs^2+3(prfs\times t)+t^2] \\ +u[3\,prfs^2+3(prfs\times u)+u^2] \end{array}$$

157. Number and Location of the Products.

In cubing a number by what are the units multiplied?

With what figure of the power will the units' product evidently end?

By what are the tens multiplied?

In multiplying $tens^2$ by tens how many times is the tens' figure taken as a factor?

How does the order of each of these factors compare with the order of the corresponding factor in the units' prodnct?

How, then, will the order of the tens' product compare with the order of the units' product?

With what figure of the power, then, will the right-hand figure of the tens' product end?

By similar reasoning explain with what figure of the cube the product by hundreds will end.

The product by thousands.

The product by tenths.

The product by hundredths.

The product by thousandths.

If, then, a dot is placed over the unit figure of a power and over each third figure at its left and right, each dot will represent the position of what?

How, then, will the number of dots compare with the number of products in the power?

With the number of figures in the root?

Give, then, a rule for finding the number of figures in the root of a given cube.

Summary of Principles. The process of extracting the cube root of a number is made evident by the relations that we have discovered between a cube and its root. For convenience of reference, however, we here summarize the

facts and principles on which the several steps of the process are based.

1. In cubing a number each figure is multiplied by three times the square of the preceding figures plus three times the preceding figures multiplied by itself plus the square of itself. Every cube, therefore, may be thought of as composed of a certain number of products, this number being equal to the number of figures in its root.

2. The right-hand product of a cube ends with the right-hand figure of the cube, and each of the remaining products ends three figures at the left of the adjacent right-hand product.

3. A zero must be annexed to the preceding figures in each product that they may be of the same order as the common factor of the last three terms.

EXPLANATION.

We are to extract the cube root of 307546.875.

We first place a dot over the 6 units and over each third figure at the left and at the right of the 6 units.

We thus find that the root of 307546.875 will consist of 3 figures, and that these figures will be tens, units, and tenths. An expression, therefore, of the relation between 307546.875 and its cube root will be $t^3 + u\,[3\text{ prfs}^2 + 3(\text{prfs} \times u) + u^2] + .t\,[3\text{ prfs}^2 + 3(\text{prfs} \times .t) + .t^2]$.

SOLUTION.

$$307\dot{5}46.87\dot{5} = t^3 + u\,[3\text{ prfs}^2 + 3(\text{prfs} \times u) + u^2] + .t\,[3\text{ prfs}^2 + 3(\text{prfs} \times .t) + .t^2]$$

	307546.875(675
	216
10800	91546
1260	
49	
12109	84763
1346700	6783875
10050	
25	
1356775	6783875

We have learned that the cube of the left-hand figure of the root will be found in the left-hand period of the power. But 307, the left-hand period of the power, is not a perfect cube. The cube of the tens, therefore, must be the greatest cube less than 307, or 216, and the tens themselves must be the cube root of 216, or 6.

Subtracting the cube of 6 from 307, and annexing to the remainder the second period of the cube, we have as a total remainder 91546.

The greater part of this remainder is made up of two factors, the first factor being the units and the second factor three times the square of the preceding figures plus three times the preceding figures multiplied by the units plus the square of the units. The

greater part of the second factor is three times the square of the preceding figures. Therefore, if we divide 91546 by three times the square of the preceding figures we shall obtain as the quotient the units or a number a little larger than the units.

Three times the square of the preceding figures, with a 0 annexed to reduce them to the order of the units, are 10800. 10800 is contained in 91546 9 times. It is evident, however, that the complete divisor that would be obtained by adding to the trial divisor three times the preceding figures multiplied by the units, etc., would not be contained in 91546 9 times. It is probable, moreover, that it would not be contained 8 times. We therefore decide upon 7 as the probable second figure of the root.

The second factor of the product [$3\,\text{prfs}^2 + 3\,(\text{prfs} \times \text{u}) + \text{u}^2$] consists not alone of three times the square of the preceding figures, but also of three times the preceding figures multiplied by the units and the square of the units. Therefore, our total second factor, or divisor, will be 10800 plus 1260 plus 49, or 12109. Multiplying this divisor by the 7 units and subtracting the product from 91546, we have as a remainder 6783. Annexing to this remainder the third period of the square we have as a total remainder 6783875.

The greater part of this remainder is made up of two factors, etc.

[Let the pupil explain in full the process of finding the tenths figure of the root, etc.]

NOTE 1. The notes given under square root will apply, with the proper modifications, to the extraction of cube root.

NOTE 2. Observe that the sum of the second and the third terms of the second factor of a product may be greater than the first term. Thus, if the first figure of a root is 1 and the second figure 9, three times the square of the preceding figures will be only 300, while three times the preceding figures multiplied by the units plus the square of the units will be 270 plus 81, or 351.

Ex. 157.

1. $\sqrt[3]{3375} = ?$
2. $\sqrt[3]{50653} = ?$
3. $\sqrt[3]{74088} = ?$
4. $\sqrt[3]{21952} = ?$
5. $\sqrt[3]{592604} = ?$
6. $\sqrt[3]{373248} = ?$
7. $\sqrt[3]{456533} = ?$
8. $\sqrt[3]{262144} = ?$
9. $\sqrt[3]{175616} = ?$
10. $\sqrt[3]{1953125} = ?$
11. $\sqrt[3]{145531576} = ?$
12. $\sqrt[3]{11239424} = ?$
13. $\sqrt[3]{6434856} = ?$
14. $\sqrt[3]{2985984} = ?$
15. $\sqrt[3]{71991296} = ?$
16. $\sqrt[3]{152273304} = ?$
17. $\sqrt[3]{277167808} = ?$
18. $\sqrt[3]{519718464} = ?$

Ex. 158.

Extract to two decimal places the cube root of each of the following numbers:

1. 567	11. 2574	21. 7856
2. 824	12. 8766	22. 785.6
3. 376	13. 5934	23. 78.56
4. 932	14. 35837	24. .9345
5. 587	15. 46925	25. .0678
6. 673	16. 72807	26. 59.34
7. 945	17. 392875	27. 937.8
8. 162	18. 784037	28. 6.327
9. 837	19. 425508	29. .0078
10. 449	20. 563784	30. 347.9

Ex. 159.

1. $\sqrt{\frac{25}{36}}=?$	11. $\sqrt{\frac{7}{25}}=?$	21. $\sqrt[3]{\frac{7}{8}}=?$
2. $\sqrt{\frac{49}{64}}=?$	12. $\sqrt{\frac{15}{32}}=?$	22. $\sqrt{\frac{22}{25}}=?$
3. $\sqrt[3]{\frac{64}{343}}=?$	13. $\sqrt{\frac{11}{13}}=?$	23. $\sqrt[3]{\frac{7}{9}}=?$
4. $\sqrt[3]{\frac{125}{216}}=?$	14. $\sqrt{\frac{3}{5}}=?$	24. $\sqrt[3]{\frac{15}{17}}=?$
5. $\sqrt{\frac{144}{169}}=?$	15. $\sqrt[3]{\frac{2}{3}}=?$	25. $\sqrt[3]{\frac{7}{32}}=?$
6. $\sqrt[3]{\frac{512}{729}}=?$	16. $\sqrt[3]{\frac{13}{17}}=?$	26. $\sqrt[3]{\frac{5}{9}}=?$
7. $\sqrt[3]{\frac{1156}{1849}}=?$	17. $\sqrt[3]{\frac{22}{35}}=?$	27. $\sqrt[3]{\frac{33}{40}}=?$
8. $\sqrt[3]{\frac{4096}{32768}}=?$	18. $\sqrt[3]{\frac{18}{19}}=?$	28. $\sqrt[3]{\frac{37}{64}}=?$
9. $\sqrt[3]{\frac{55696}{1823329}}=?$	19. $\sqrt[3]{\frac{7}{18}}=?$	29. $\sqrt[3]{\frac{116}{250}}=?$
10. $\sqrt{\frac{2000376}{1658137}}=?$	20. $\sqrt{\frac{13}{32}}=?$	30. $\sqrt{\frac{35}{81}}=?$

NOTE. It may be advisable to multiply both terms of a fraction by such a number as will make the denominator a perfect power. Thus, the square root of $\frac{15}{32}$ equals the square root of $\frac{30}{64}$, and the cube root of $\frac{7}{9}$ equals the cube root of $\frac{21}{27}$.

PROBLEMS.

1. The surface of the planet Mercury is .14 that of the Earth. What is the ratio of Mercury's diameter to the diameter of the Earth?
2. The volume of the planet Venus is .92 that of the Earth. What is the ratio of the diameter of Venus to the diameter of the Earth?
3. The surface of the planet Mars is .29 that of the Earth. The diameter of the Earth is 7912 miles. What is the diameter of Mars?
4. The volume of the planet Jupiter is 1309 times that of the Earth. What is the diameter of Jupiter?
5. The ratio of the surface of the planet Saturn to the surface of Jupiter is .71. What is the diameter of Saturn?

6. The ratio of the volume of the planet Uranus to the volume of Jupiter is .05. What is the ratio of the diameter of Uranus to the diameter of Jupiter?

7. The ratio of the surface of the planet Neptune to the surface of Jupiter is .16. What is the ratio of the diameter of Neptune to the diameter of Jupiter?

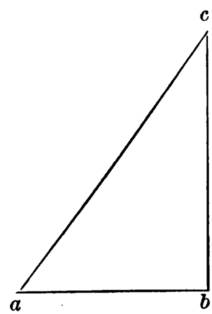
8. The volume of the Sun is 6,550,000 times the volume of the Moon. What is the ratio of the diameter of the Sun to diameter of the Moon?

9. The value of a sphere of gold with a certain diameter is $1,000. Find the ratio of the diameter of this sphere to the diameter of a sphere worth $5,000.

10. Through a certain pipe only ⅛ the required quantity of water can pass. Find the diameter of the pipe from which the required quantity can be secured.

158. Relations of the Sides of a Right Triangle.

A triangle containing a right angle is called a **Right Triangle**. That side of a right triangle which is opposite the right angle is called the **Hypothenuse**; that one of the remaining sides on which the triangle is supposed to stand is called the **Base**; and that side which is perpendicular to the base is called the **Perpendicular**.



What kind of a triangle is the triangle *a b c*?
Which side is the hypothenuse?
The base?
The perpendicular?

Measure the base *a b*. Its length is how many thirds of an inch?
Measure the perpendicular *b c*. Its length is how many thirds of an inch?
Measure the hypothenuse *a c*. Its length is how many thirds of an inch?

3^2 equals what?
4^2 equals what?
3^2 plus 4^2 equals what? 5^2 equals what?
How, then, does the square of the hypothenuse of the triangle *a b c* compare with the squares of the other two sides?

By geometry it may be shown that the same relation exists between the sides of every right triangle. Hence the following principle:

In a right triangle the square of the hypothenuse equals the square of the perpendicular plus the square of the base.

* *

The base of a right triangle is 5 feet and the perpendicular 12 feet.

What is the square of the base? of the perpendicular?

What, then, is the square of the hypothenuse?

What is the hypothenuse?

Give a rule for finding the hypothenuse of a right triangle when the base and the perpendicular are given.

The hypothenuse of a right triangle is 17 feet and the base 8 feet.

What is the square of the hypothenuse? of the base?

What, then, is the square of the perpendicular?

What is the perpendicular?

Give a rule for finding the perpendicular of a right triangle when the hypothenuse and the base are given.

The hypothenuse of a right triangle is 5 feet and the perpendicular 4 feet.

What is the square of the hypothenuse?

What is the square of the perpendicular?

What, then, is the square of the base?

What is the base?

Give a rule for finding the base of a right triangle when the hypothenuse and the perpendicular are given.

Ex. 1. The base of a right triangle is 15 feet and the perpendicular 25 feet. What is the hypothenuse?

Ex. 2. The base of a right triangle is 30 feet and the hypothenuse 40 feet. What is the perpendicular?

SOLUTIONS.

(1)	(2)
225	1600
625	900
850	700
$\sqrt{850}$	$\sqrt{700}$
$=29+$	$=26+$

EXPLANATIONS.

Ex. 1. The square of the hypothenuse of a right triangle equals the square of the base plus the square of the perpendicular. In the given square, therefore, the square of the hypothenuse equals $15^2 + 25^2$, or $225+625$, or 850; and the hypothenuse equals the square root of 850, or $29+$.

Ex. 2. The square of the hypothenuse of a right triangle equals the square of the base plus the square of the perpendicular. The square of the perpendicular, therefore, equals the square of the hypothenuse minus the square of the base, or $40^2 - 30^2$, or $1600 - 900$. or 700; and the perpendicular equals the square root of 700, or $26+$.

Problems.

1. The base of a right triangle is 14 feet and the perpendicular 18 feet. What is the hypothenuse?
2. The perpendicular of a right triangle is 9 inches and the hypothenuse 11 inches. What is the base?
3. The base of a right triangle is 27 feet and the hypothenuse 42 feet. What is the perpendicular?
4. The base of a right triangle is 7 yards and the hypothenuse 13 yards. What is the perpendicular in yards, feet, and inches?
5. The base of a right triangle is 37 rods and the hypothenuse 54 rods. What is the perpendicular?
6. A room is 15 feet long and 12 feet wide. Find the distance between its opposite corners.
7. A room is 20 feet long and the distance between its opposite corners is 25 feet. Find the width of the room.
8. A room is 16 ft. 10 in. long, 14 ft. 6 in. wide, and 8 ft. 4 in. high. Find the distance from a lower corner to the opposite upper corner.
9. A regulation base-ball ground is laid out in the form of a square, the distance between any two adjacent bases being 90 feet. Find the distance from home-base to second-base.

159. Area of an Inscribed Square.

Draw a circle. Draw one of its diameters and connect the central point of one of the semi-circumferences thus formed with the ends of the diameter. What kind of a triangle do you thus form? Which line is the hypothenuse? Which lines are the base and the perpendicular?

What relation, then, has the sum of the squares of the two shorter lines to the square of the diameter?

How do the two shorter lines compare with each other in length?

What relation, then, has the square of either of the shorter lines to the square of the diameter?

Connect the central point of the remaining semi-circumference with the ends of the diameter. What figure do you form?

What figure is formed by these two lines and the two corresponding lines in the first semi-circle? What relation has the area of this figure to the square of either of the four sides?

What relation, then, has it to the square of the diameter?

Give, then, a rule for finding the area of the square that can be inscribed in a given circle?

Give a rule for finding the side of the square that can be inscribed in a given circle?

	SOLUTIONS.	
	(1)	(2)
Ex. 1. Find the area of the square that can be inscribed in a circle whose diameter is 14 inches.	98	$36\frac{1}{8}$

Ex. 2. Find the area of the square that can be inscribed in a circle whose diameter is $8\frac{1}{2}$ inches.

EXPLANATIONS.

Ex. 1. Imagine a diameter to be drawn in the given circle, and the centres of the two semicircumferences thus formed to be connected with its ends. The figure thus formed will be a square, and its area will be the second power of any one of the four sides.

It is evident that the diameter of the circle divides the square into two equal right triangles, and that the square of the diameter is the sum of the squares of either of the two pairs of sides, or, all the sides being equal, twice the square of any side. The required area, therefore, is $14^2 \div 2$, or 14×7, or 98.

Ex. 2. Explain the solution of Ex. 2.

NOTE. Observe that the shorter solution in the second exercise is to square the $8\frac{1}{2}$ and then divide by 2. Why?

Ex. 160.

Find the area of the square that can be inscribed in a circle with

1. A diameter of 8 inches.
2. A radius of 5 inches.
3. A circumference of 10 inches.
4. A diameter of 15 inches.
5. A diameter of 27 inches.
6. A diameter of $16\frac{1}{2}$ inches.
7. A circumference of 25 inches.
8. A circumference of 40 inches.
9. A radius of 11 inches.
10. A diameter of $27\frac{1}{2}$ inches.

Vermont Rule for Measuring Logs.

The principle for finding the area of a square that can be inscribed in a given circle finds a practical application in the measurement of logs by the Vermont statute, which assumes:

1. **That the largest possible square stick of lumber is sawn from each log.**
2. **That the lumber that can be sawn from the slabs thus formed will substantially compensate for the lumber that would be converted into sawdust in sawing the rectangular stick into boards or planks.**

NOTE. The Vermont statute for the measurement of logs was enacted in 1884. The important minor principles of the law are the following:

1. The bark of the log must not be included in the diameter.

2. The diameter is to be measured at the smaller, or "top", end.

3. In measuring a diameter, a fraction of an inch less than ½ is to be rejected, and a fraction greater than ½ is to be considered an additional inch. In other words, no fraction shall be taken into account in measuring a diameter unless it is an exact half-inch.

4. The law provides for a new diameter being obtained at each interval of 12 feet from the small end. In practice, however, only one measurement is taken unless the log is of unusual length.

What do we regard as the producing surface of a log or other cylindrical body?

How do we find the area of the square that can be inscribed in a given circle?

How, then, shall we find the surface of the end of the largest square stick of lumber that can be sawn from a log of given diameter?

By what must we multiply this surface to obtain the contents of the imaginary square stick?

Give, then, a general rule for finding the contents of a log according to the Vermont statute.

In what denomination would the diameter of a log naturally be expressed?

In what denomination the length of a log?

How many, then, of the three factors used in finding the contents of a log by the Vermont statute are commonly expressed in inches? in feet?

In finding the number of board feet in a piece of lumber how many of the three dimensions are expressed in inches?

How many in feet?

By what, then, must we divide the contents obtained under the Vermont statute that the result may be in board feet?

Suppose that the log is 12 feet long. After multiplying the diameter in inches by half the diameter in inches what are the two remaining steps?

What will be the effect of first multiplying by 12 and then dividing by 12?

Give, then, a special rule for finding in board feet according to the Vermont statute the contents of a log 12 feet long.

What will be the contents as compared with a log 12 feet long of a log

14 feet long?	**11 feet long?**
15 feet long?	**18 feet long?**

Give, then, a special rule for finding by the Vermont statute the contents of a log

14 feet long.	**11 feet long**
15 feet long?	**18 feet long.**

	SOLUTIONS.		
Ex. 1. Find the contents of a log 18 inches in diameter and 12 feet long.	(1) 162	(2) 200	(8) 50°
Ex. 2. Find the contents of a log 20 inches in diameter and 11 feet long.		183	83⅓

Ex. 3. Find the contents of a log 10 inches in diameter and 20 feet long.

EXPLANATIONS.

Ex. 1. The Vermont statute provides that a log shall be considered to contain a quantity of lumber equal to the quantity contained in the largest square stick that can be sawn from the log.

The area of the square that can be inscribed in a circle with a diameter of 18 inches is 18×9 square inches. To complete the solution we should multiply this surface by 12, the length of the log, and then divide by 12 that two of the dimensions may be in feet and one in inches, as required in board measure. As, however, these operations would counterbalance each other, the required contents are 18×9, or 162, board feet.

Exs. 2 and 3. Explain the solution of these exercises.

Ex. 161.

Find by the Vermont statute the quantity of lumber in the following logs, the left-hand figures in each column representing the length in feet and the right-hand figures the diameter in inches:

12—7	12—18	12—19	14—11
12—11	12—15½	10—29½	8—2½
10—13	12—18	12—35	12—11
14—16½	12—17	11—14½	11—7½
12—6	12—10½	12—13	11—13
11—6	14—26	13—16	12—16

Art. 160. Scribner's Rule for Measuring Logs.

Logs are very generally measured by a process called **Scribner's Rule,"** which is a series of tables giving the contents of logs of different lengths and different diameters. Below are the contents of logs 12 feet in length and with diameters ranging from 8 to 35 feet taken from the 1882 edition of Scribner's Lumber and Log Book.

8—12	15—91	22—243	29—469
9—19	16—108	23—271	30—507
10—27	17—127	24—300	31—547
11—37	18—147	25—331	32—588
12—48	19—169	26—363	33—631
13—61	20—192	27—397	34—675
14—75	21—217	28—432	735—21

Find by what per cent the contents of each of the preceding logs by the Vermont statute are greater or less than the contents by the Scribner's rule.

161. New Hampshire Rule for Measuring Logs.

The following is the New Hampshire statute for the measurement of round timber:

"All round timber, the quantity of which is estimated by the thousand, shall be measured according to the following rule: a stick of timber sixteen inches in diameter and twelve inches in length shall constitute a cubic foot, and the same relation shall apply to any other ratio and quantity. Each cubic foot shall constitute 10 feet of a thousand."

How many board feet by the Vermont rule are there in a log 12 inches long and 16 inches in diameter?

How many by the New Hampshire rule?

What is the ratio of the contents by the New Hampshire rule to the contents by the Vermont rule?

10 divided by $10\frac{2}{3}$ equals what?

$\frac{15}{16}$ is how much less than $\frac{16}{16}$?

How, then, does the quantity of lumber in a log by the New Hampshire statute compare with the quantity by the Vermont statute?

Give a rule for finding in accordance with the New Hampshire statute the quantity of lumber in a given log.

	SOLUTION.
The first of three logs is 12 feet long and 15 inches in diameter; the second, 14 feet long and 8 inches in diameter; and the third, 12 feet long and $19\frac{1}{2}$ inches in diameter. Find the contents of the three logs according to the New Hampshire statute.	$112\frac{1}{2}$ 37 190 ——— 340 21 ——— 319

EXPLANATIONS.

Ex. 1. The contents of the three logs by the Vermont statute would be $112\frac{1}{2}+37+190$, or 340, board feet. A log by the New Hampshire statute measures $\frac{1}{16}$ less than by the Vermont statute, therefore the required contents will be $340 - \frac{1}{16}$ of 340, or 319, board feet.

NOTE 1. If a $\frac{1}{2}$ occurs in the contents of a single log, retain it; if a fraction less than $\frac{1}{2}$, reject it; and if a fraction more than $\frac{1}{2}$ call it an integer.

NOTE 2. It is evident that the contents of a log according to the New Hampshire statute can also be found by the following process:

1. Find the ratio of the given diameter to 16. Why?
2. Square this ratio. Why?
3. Multiply this square by the length of the log. Why?

Ex. 162.

Find according to the New Hampshire statute the quantity of lumber in the logs whose dimensions are given in the following columns. In these columns, the figures at the top, which are enclosed in parentheses, represent the lengths of the logs in feet and the remaining figures the diameters in inches.

(12)	(12)	(14)	(10)	(11)	(13)	(13)	(9)	(24)	(18)
18	12	25	24	37	17	21	18	16	38
13½	8	15	13	24	20½	17	22	19	7
15	18	7½	12	6	14	20	36	18	9
17	31½	27	35	15	31½	23½	24	13	4
11½	14	29	17½	18½	28	39	25	28½	25
15	16	17	16	18	15	16	14½	13	31
10½	8½	7½	13	24	20½	22	23	24	25
8	16	13	18	29	33	34	15	16	27

Problems.

Ths largest tree known to exist in the world has been recently discovered on a government reservation in Fresno County, California. The tree is said to be 350 feet high and 51 feet in diameter 6 feet from the ground.

In the following problems assume that the tree tapers uniformly from the base to the top.

1. How many logs 6 inches or more in diameter at the smaller end could be cut from the tree, and what would be the diameter of each log?

2. How many board feet of lumber according to the Vermont rule are there in the tree?

3. How many board feet of lumber are there in the tree according to the New Hampshire rnle?

4. How many board feet would there be in the entire tree if it could be converted into lumber without waste in the form of sawdust and slabs?

5. What would be the value, according to each of the three measurements, of the lumber from the tree at $25 a thousand?

6. Find the weight of the tree, assuming its specific gravity to be .48, the specific gravity of spruce.

PROGRESSIONS.

One of the contests in athletic sports is a potato race. In this contest the potatoes are placed in a straight line at a certain common distance from each other, and the contestants are required to pick the potatoes up and place them in a basket, making a separate trip for each potato.

Suppose that the basket is placed 10 feet from the first potato, and that the potatoes are placed 3 feet apart.
What is the distance from the basket to the second potato? to the third potato? to the tenth potato? to any potato?

162. Arithmetical Progressions.

The distances from the basket to the several potatoes form a series of numbers each of which is greater than the preceding by three. Such a series of numbers is called an **Arithmetical Progression**, and the difference between any two consecutive terms is called the **Common Difference.**

How in the potato contest did we find the distance from the basket to any potato?
Give, then, a rule for finding any term af an arithmetical progression when the first term, the common difference, and the number of the term are given.
Suppose that the series were a decreasing series. How should we modify this rule?

The distance from the basket to a certain potato is 40 feet. We wish to find the number of the potato.
Of what two parts is the distance between this potato and the basket made up?
The first part is how many feet?
The second part, then, is how many feet?

Of what two factors is the second part made up?
What is the common difference?
What, then, is the number of the term less 1?
What is the number of the term?
Give a rule for finding the number of the term when the term itself, the first term, and the common difference are given.

In a second contest the basket is 15 feet from the first potato. The fourteenth potato is 80 feet from the first. We wish to find the common distance between the potatoes.
Of what two parts is the distance between the fourteenth potato and the basket made up?
The first part is how many feet?
The second part, then, is how many feet?

Of what factors is this second part made up?
What is the number of the given potato? the number less 1?
What, then, is the common difference?
Give a rule for finding the common difference when the first term, the last term, and the number of terms are given.

We wish to find how far the contestants must travel in the first contest to pick up five potatoes.

10	13	16	19	22
22	19	16	13	10

To do this we first write out the series 10, 13, 16, etc., to represent the distance they travel in going to the several potatoes, and the same series underneath in reverse order to represent the distance they travel in returning The next step is to combine the corresponding terms of these series.
What is the sum of each pair of these terms?
How does this sum compare with the sum of the first and the last terms of the series?

How does the number of pairs compare with the number of terms?
How, then, without writing the intermediate terms, might we have found the sum of the two series?
How, evidently, after finding the sum of the two equal series, can we find the sum of either one?
Give, then, a rule for finding the sum of an arithmetical series when the first and the last terms and the number of terms are given.

SOLUTIONS.

(1)	(2)	(3)	(4)
56	33	40	100
	4	10	400

Ex. 1. The first term of an arithmetical series is 20, and the common difference 6. What is the seventh term?

Ex. 2. The first term of an arithmetical series is 14, the last term 47, and the common difference 11. What is the number of the last term?

Ex. 3. The first term of an arithmetical series is 35, and the fifth term 75. What is the common difference?

Ex. 4. The first term of an arithmetical series is 25, the last 75, and the number of terms 8. What is the sum of the series?

EXPLANATIONS.

Ex. 1. The second term of the series is 20, the first term, plus 6, the common difference; the third term is 20, the first term, plus 12, the common difference taken twice. This law will hold for any number of terms, therefore the seventh term is 20, the first term, plus 36, the product of the common difference by the number of the term less 1, or 56.

Ex. 2. 47, the last term of the given series, is made up of two parts, 14, the first term, and the product of 11, the common differ-

ence, by the number of terms less 1. This product, therefore, equals $47 - 14$, or 33; the number of terms less 1, the required factor, equals 33 divided by 11, the common difference, or 3; and the number of terms equals $3 + 1$, or 4.

Ex. 3. Explain the solution of Ex. 3.

Ex. 4. If we should write this series and then write it again underneath in reverse order the sum of each pair of terms would be the same and would be the sum of the first and the last terms, or $25 + 75$, or 100. The number of the pairs, moreover, would be the number of the terms, or 8. Therefore, the sum of the two series would be 100×8, and the sum of either series, $(100 \times 8) \div 2$, or 400.

Ex. 163.

In the following exercises a represents the first term, l the last term, d the common difference, s the sum, and n the number of the term or the number of terms.

Find the missing term in each exercise.

	a	l	d	n	s		a	l	d	n	s
1.	3	( )	5	4		6.	47	( )	5	9	
2.	5	45	4	( )		7.	36	100	10	( )	
3.	11	56		6	( )	8.	( )	225	12	11	
4.	10	62	4	( )		9.	26	229	29	( )	
5.	( )	75	7	11		10.	6	97	14		( )

163. Geometrical Progression.

A corporation invests $100,000 so that it doubles itself every ten years. What will be the amount of the investment at the beginning of the second period of ten years? at the beginning of the third period? by what power of the ratio must we multiply the first term to find the amount of the investment at the beginning of the tenth period? of any period?

The amounts of the investments at the beginning of the several periods form a series of numbers each of which is twice the preceding. Such a series is called a Geometrical Progression, and the ratio of any term to the preceding is called the Ratio of the series.

How in the investment problem did we find the amount of the investment at the beginning of any period?

Give, then, a rule for finding any term of a geometrical series when the first term, the ratio, and the number of the term are given. Suppose that the series were a decreasing series how should we modify this rule?

The amount of the investment at the beginning of a certain period is $1,600,000. We wish to find the number of the period.

Of what two factors is the $1,600,000 the product?
The first term is how many dollars?
2 raised to a power one less than the number of the term is, then, what number?

To what power must 2 be raised to produce 16?
What, then, is the number of the term less 1?
What is the number of the term?
Give, then, a rule for finding the number of a term of a geometrical series when the term itself, the first term, and the ratio are given.

* *

In a second investment of $1,000 the amount at the beginning of the fifth period of ten years was $81,000. We wish to find the ratio by which the investment increases.

Of what two factors is the $81,000 the product?
The first term is how many dollars?
The ratio raised to a power one less than the number of the term is, then, what number?
What is the number of the term? the number less 1?
What number raised to the fourth power will produce 81?
What, then, is the required ratio?
Give a rule for finding the ratio of a geometrical series when the first term, the last term, and the number of the last term are given.

A laborer	3	9	27	81	243	729	2187	6561	19683	59049
is hired for	1	3	9	27	81	243	729	2187	6561	19683

ten months upon the agreement that he receive one cent for the first month, three cents for the second month, nine cents for the third month, and so on to the tenth month. We wish to find the total amount he will receive.

What series represents the amount he will receive?
By what do we multiply this series?
What second series do we obtain?
How does this series compare with the first?

The next step is to subtract the first series from the second. By what single operation can we perform this subtraction?
59049 less 1 equals what?
Three times the series less once the series are how many times the series?
Twice the series, then, is what number?
Once the series is what number?

Suppose that we had not written out the second series. How might we have obtained its last term?
How, then, without writing out the intermediate terms, could we find the sum of the given series?
Give a rule for finding the sum of a geometrical series when the first and the last terms and the ratio are given.

	SOLUTIONS.			
Ex. 1. The first term of a geometrical series is 7, and the ratio 5. What is the fifth term?	625	216	256	187500
	4375	6	5	187488
				46872

Ex. 2. The first term of a geometrical series is 10, and the fourth term 2160. What is the ratio?

Ex. 3. The first term of a geometrical series is 9, the last term 2304, and the ratio 4. What is the number of terms?

Ex. 4. The first term of a geometrical series is 12, the last term 37500, and the ratio 5. What is the sum of the series?

EXPLANATIONS.

Ex. 1. The second term of the given series is 7, the first term, multiplied by the ratio; the third term is 7, the first term, multiplied by the ratio raised to the second power; the fourth term is the first term multiplied by the ratio raised to the third power. This law, evidently, will hold for any number of terms. The fifth term, therefore, is the first term multiplied by the ratio raised to a power one less than the number of the term, or 7×5^4, or 7×625, or 4375.

Ex. 2. The last term of a geometrical series equals the first term multiplied by the ratio raised to a power one less than the number of the term. In the given series, therefore, the ratio raised to the third power equals 2160, the last term, divided by 10, the first term, or 216, and the ratio equals the third root of 216, or 6.

Ex. 3. Explain the solution of Ex. 3.

Ex. 4. If we should write out this series and multiply it by the ratio we should obtain a second series identical with the first except that the first term of the first series would not appear and the last term would equal the last term of the first series multiplied by the ratio. Therefore, to subtract once the series from the ratio times the series we simply subtract 12, the first term, from 187,500, the last term multiplied by the ratio. 187,488, the remainder we thus obtain, is 5 — 1, or 4, times the series, and the series itself is 187,488, divided by 4, or 46,872.

Ex. 164.

In the following exercises a represents the first term, l the last term, r the satio, s the sum, and n the number of the term or the number of terms.

Find the missing term in each exercise.

	a	l	r	n	s		a	l	r	n	s
1.	8	( )	4	5		6.	74	( )	9	5	
2.	5	135	3	( )		7.	15	960	8	( )	
3.	11	6875		5	( )	8.	( )	6144	4	6	
4.	20	2420	11	( )		9.	14	3784	3	( )	
5.	( )	1715	7	4		10.	6	6144	2		( )

Problems.

1. A boy learning to ride a bicycle rides 3 miles the first day and increases this distance daily by 2 miles until he rides 15 miles a day. On what day does he first ride the 15 miles? What is the total distance that he has ridden at the end of this day?

2. A second boy rides 3 miles the first day and increases this distance each day so that on the ninth day he rides 15 miles. What is his daily increase?

3. The population of Vermont in 1880 was 332,286 and in 1890 332,452. What was the annual increase in population?

4. If the annual increase during the ten years had been one per cent what would have been the population of Vermont in 1890?

5. Find the amount at compound interest of $250 for 10 years at 6 per cent.

6. What principal at compound interest will amount to $506 in 4 years at 6 per cent?

7. At what rate will $300 at compound interest amount to $357.3048 in 3 years?

8. In how many years will $750 at 7 per cent compound interest amount to $918.78225?

9. Suppose that in the potato race described in the inductive exercise one contestant begins with the nearest potato and works from the basket and the second with the fifteenth potato and works toward the basket. When the first has put 5 in the basket how many must the second have put in that he may be ahead of the first?

10. A body under the influence of gravity falls 16.1 feet the first second and gains in velocity each second 32.2 feet. How many feet will it fall the tenth second? how many in ten seconds?

11. The population of Vermont in 1900 was 343,641. Find its average annual increase from 1890 to 1900.

12. How many strokes does an ordinary clock strike in 24 hours?

13. A laborer wrought 20 days, and received for the first day's labor 4 grains of rye, for the second 12, for the third 36, &c. How much did his wages amount to, allowing 7680 grains to make a pint, and the whole to be disposed of at $1 per bushel?—*Burnham's Arithmetic, Edition of 1841.*

Interest. Money paid for the use of money.

Principal. The money loaned.

Simple Interest. Interest that when unpaid draws no interest.

Annual Interest. Interest that when unpaid draws simple interest.

Compound Interest. Interest that when unpaid is made a part of the principal.

Percentage. The process of finding a product or a factor when the given or the required ratio is expressed or is to be expressed in hundredths.

Corporation. A collection of persons given existence, and legal privileges and obligations, as a single body.

Charter. A certificate of incorporation,

Stock. The capital invested in the corporation.

Share. The unit of investment.

Public Corporations. Corporations formed for the government and welfare of all the people living within the territory covered by it.

Private Corporations. Corporations formed by persons who voluntarily become members either for private gain or for charitable, educational, or religious purposes.

Quasi-Public Corporations. Private corporations supposed to be essential to the public welfare that are given certain special privileges and are subject to certain special restrictions.

Note. An unqualified written promise to pay a definite sum of money at a specified time or on demand.

Bond. A note under seal.

Exchange. The purchasing of a sum of money to be paid at a specified place to a specified person.

Taxes. Money paid for the support of government.

Insurance. An agreement for a specified compensation to pay a specified sum upon the occurrence of a specified event.

Commission. A certain per cent paid a person for buying or selling the goods of another person.

Equated Time. The time at which several debts due at different times can be justly paid.

Antecedent. The first term of a ratio.

Consequent. The second term of a ratio.

Extremes. The first and the last terms of a proportion.

Means. The second and the third terms of a proportion.

Partnership. An association of persons in business of a less permanent character than a corporation.

Arithmetical Progression. A series of numbers increasing or decreasing by a common difference.

Geometrical Progression. A series of numbers increasing or decreasing by a common ratio.

REVIEW QUESTIONS.

Define interest; principal; rate; amount.

What is the interest on $1 at 6% for 1 year? for 2 months? for 6 days? for 1 month? for 1 day?

Give a rule for finding the interest on any principal for 60 days; for any convenient part of 60 days; for 6 days; for any convenient part of 6 days.

Give a rule for writing the interest at 6% on $1 for years, months, and days; for finding the interest at 6% on any principal for years, months ,and days; for finding the interest at any rate on any principal for years months and days, for months and days, for any number of days.

Define annual interest; direct interest; indirect interest. Explain the process of finding the time for the indirect interest.

Give a rule for computing annual interest.

Define compound interest. Give a rule for computing compound interest; for computing compound interest from a compound interest table.

Give a rule for finding the principal when the interest, time, and rate are given; for finding the time when the interest, principal, and rate are given; for finding the rate when the interest, principal, and time are given.

Give the Merchant's Rule for partial payments. When is this rule used? Give the two fundamental principles of the United States Rule. Where is this rule used?

Give the principles of the Vermont Rule for partial payments; of the New Hampshire Rule.

Define percentage; base; rate. What are the three cases of percentage? Give a rule for each case.

What are the four cases in profit and loss? Give a rule for each case. Give a rule for finding the net price of an article when several trade discounts are given.

Define corporation; charter; shares; stock; stockholder; stock certificate; dividend; assessment; public corporation; private corporation; note; bond.

When are stocks at par? above par? below par?

Give a general principle for determining the market value of stocks.

Upon what three conditions does the market value of a bond depend?

Give four cases in problems in stocks. Give a rule for each case.

Define equation of payments. Give a rule for finding a date at which to pay several debts due at different dates; for finding a date at which to pay the balance of a debt when part is paid before it is due.

Give four cases in commission and brokerage. Give a rule for each case.

Define banking; banks of issue; savings banks; private banks. Give the principal conditions essential to the formation of a National Bank. By whom are deposits in National Banks principally made? in savings banks? On what security is money commonly loaned in National Banks? in savings banks?

What is said as to the character of private banks and as to the regulations controlling them?

Give the characteristic features of the stock certificate given near the end of the book; of the bond; of the first note; of the second; of the third; of the fourth; of the check.

Define indorsement; explain the different forms of indorsement of a note, of a check.

Explain the process of computing time on a note given to a bank. Give a rule for finding the face of a note when the proceeds are given. Explain the process of computing interest on deposits in the Sugar River Savings Bank.

What are the three principal agencies of money exchange? Explain the different ways of transmitting money through the mail; through express companies; through banks.

Define taxes; customs. What are the different classes of taxes in most States on the basis of the use to which they are applied? on the basis of the source from which they are derived?

Define present worth; true discount.

Give a rule for finding present worth; for finding true discount.

Give a rule for finding the missing term of a proportion when the other three terms are given. Define an antecedent; a consequent; a compound ratio: a compound proportion; an inverse ratio; an inverse proportion.

Give a rule for solving a problem by proportion.

Define partnership. Give a rule for the division of the gain or

loss when each partner's capital is invested for the same time; when the capital of each is invested for different times.

Define the principal terms used in involution; in evolution.

Define a factor; a term; a monomial; a binomial; a trinomial.

Square the binomial $(20+3)$; the binomial $(\text{prfs}+\text{u})$. Factor the last two terms of the square of each binomial.

By what are the units multiplied in squaring a number? by what the tens? the tenths? by what is any figure multiplied?

Explain the finding of the number and location of the products in a square. Give a rule for the extraction of the square root of a number.

Raise the binomial $(\text{prfs}+\text{u})$ to the third power. Remove the common binomial factor from the last three terms of the power.

By what are the units multiplied in cubing a number? by what the tens? the tenths? By what is any figure multiplied?

Explain the finding of the number and location of the products in a cube. Give a rule for the extraction of cube root.

Give a rule for extracting the square root of a fraction; the cube root.

Define a right triangle; the hypothenuse; the base; the perpendicular.

What relation has the square of the hypothenuse to the square of the other two sides?

Illustrate.

Give a rule for finding the hypothenuse when the other two sides are given; for finding either side when the hypothenuse and the other side are given.

Give a rule for finding the area of a square that can be inscribed in a circle of given diameter. Show upon what principles the rule is based.

Upon what assumption is the Vermont statute for measuring logs based? Upon what principle is the process of finding the surface of the end of the imaginary stick based?

Give a general rule for finding the contents of a log by the Vermont statute; a special rule for finding the contents in board feet of a log 12 feet in length; a general rule based upon this special rule for finding in board feet the contents of logs of other lengths than 12.

What is said concerning Scribner's Rule for measuring logs?

Give the substance of the New Hampshire statute for the measurement of logs? How do the contents of a log by the New Hampshire statute compare with the contents by the Vermont statute?

Define an arithmetical progression; a geometrical progression; a common difference; the ratio of a series.

Make, solve, and explain a problem in each of the four given cases in arithmetical progressions; in geometrical progressions. Give a rule for each case.

Define an increasing series; a decreasing series.

What problems in compound interest can be solved as problems in geometrical progressions. Make, solve and explain a problem of each class.

Review Problems.

1. Find the simple interest at 6 per cent on $296.34 for 60 day; on $219.36 for 6 days; on $104.28 for 10 days; on $75.98 for 4 days; on $407.28 for 45 days; on $96.25 for 116 days.

2. Find the simple interest at 6 per cent on $1 for 2 yr. 5 mo. 16 da.; on $528.14 for 4 yr. 6 mo. 17 da.; on $908.24 for 3 yr. 11 mo. 28 da.

3. Find the simple interest at 6 per cent on $1000 for 4 yr. 9 mo. 19 da.; on $25 for 2 yr. 2 mo. 26 da.; on $99.75 for 3 yr. 11 mo. 22 da.; on 37 for 25 days.

4. Find the simple interest on $382.47 for 4 yr. 1 mo. 10 da. at 8 per cent; on $397 for 2 yr. 10 mo. 13 da. at 10 per cent; on $600 for 2 yr. 5 mo. 4 da. at 7 per cent; on $216.25 for 3 yr. 9 mo. 17 da. at 8½ per cent.

5. Find the simple interest on $95.28 for 48 days at 5 per cent; on $134.26 for 56 days at 9 per cent; on $60 for 45 days at 7 per cent.

6. Find the interest at 6 per cent on a note of $275 from Jan. 16, 1902 to June 5, 1902. Find the exact interest on the note; the bank discount.

7. Find the annual interest at 6 per cent on $234.76 for 3 yr. 7 mo. 21 da.; on $95 at 5 per cent from June 2, 1901 to March 8, 1904.

8. Find the compound interest, payable semi-annually, on $193.78 for 3 yr. 5 mo, 23 da.; on $400 for 25 yr. 9 mo. 28 da.

9. A note bearing annual interest at 6 per cent for $456.28 is given Aug. 24, 1900. The following payments are made on the note: Jan. 15, 1901, $12.75; July 1, 1902, $140; and Feb. 20, 1903, $75. Find the amount due on the note July 1, 1903 according to the Vermont Rule; according to the New Hampshire Rule; according to the United States Rule.

Find the amount due on the preceding note if computed according to the Merchants' Rule (a) by simple interest, (b) by annual interest, (c) by compound interest.

10. Make and solve 3 problems in profit and loss; 4 in stocks and bonds; 3 in bank discount; 2 in exchange: 2 in insurance; 2 in taxes; 2 in equation of payments; 2 in commission and brokerage; 2 in proportion; 2 in partnership; 4 in arithmetical progressions; 4 in geometrical progressions.

11. Extract the square root of 21418384; of 29506624: the cube root of 12326391; the square root to three decimal places of 5; of 1.6; of $\frac{1}{2}$; the cube root to three decimal places of 9; of .27; of $\frac{3}{8}$.

STOCK CERTIFICATE.

No. .... CERTIFICATE OF STOCK .... Shares.

OF

THE HILLSIDE CREAMERY,

OF CORNISH, N. H.

THIS IS TO CERTIFY, that .... is the owner of .... shares of twenty-five dollars each, of the Capital Stock of the Hillside Creamery of Cornish, N. H., transferable only on the books of the company in person, or by attorney, or the surrender of this certificate.

In witness whereof, the said Hillside Creamery has caused this certificate to be signed by its President and Treasurer, at Cornish, N. H., this .... day of .... A. D. 19

.... Treasurer. .... President.

Bank Check.

No. .... *Concord. N. H.,* *190*

NATIONAL STATE CAPITAL BANK
of Concord, N. H.

Pay to the order of .... $\frac{}{100}$ **$**.................

....***Dollars.***

....

Forms of Indorsement.

(1)

J. F. Smith.

(2)

Pay to order of Frank Brown.
J. F. Smith.

(3)

Without Recourse,
J. F. Smith.

(4)

Waiving Protest and Notice,
J. F. Smith.

Contents of Parts 3 and 4.